LECTURE 12: Aperture Antennas – Part I

(*The uniqueness theorem. The equivalence principle. The application of the equivalence principle to aperture problem. The uniform rectangular aperture. The tapered rectangular aperture.*)

Introduction

Aperture antennas constitute a large class of antennas, which emit electromagnetic wave through an opening (or aperture). These antennas have close analogs in acoustics: the megaphone and the parabolic microphone. The pupil of the human eye is a typical aperture receiver for optical EM radiation. At radio and microwave frequencies, horns, waveguide apertures and reflectors are examples of aperture antennas. Aperture antennas are of common use at UHF and above. It is because aperture antennas have their gain increase as $\sim f^2$. For an aperture antenna to be efficient and have high directivity, it has to have an area comparable or larger than λ^2. Obviously, these antennas would be impractical at low frequencies. Another positive feature of the aperture antennas is their near-real valued input impedance and geometry compatibility with waveguide feeds.

To facilitate the analysis of these antennas, the equivalence principle is applied. This allows us to carry out the far-field analysis in the outer (unbounded) region only, which is external to the radiating aperture and the antenna. This requires the knowledge of the tangential field components at the aperture, as it follows from the equivalence principle.

1. Uniqueness theorem

A solution is said to be unique if it is the only one possible among a given class of solutions.

The EM field in a given region $V_{[S]}$ is uniquely defined if

- *all sources are given;*
- *<u>either</u> the tangential $\vec{E}_\tau$ components <u>or</u> the tangential $\vec{H}_\tau$ components are specified at the boundary S.*

The uniqueness theorem is proven by making use of the Poynting's theorem in integral form:

$$\oiint_S \left(\vec{E}\times\vec{H}^*\right)d\vec{s} + j\omega\iiint_{V_{[S]}}\left(\mu\,|\,\vec{H}\,|^2 - \varepsilon\,|\,\vec{E}\,|^2\right)dv + \iiint_{V_{[S]}}\sigma\,|\,\vec{E}\,|^2\,dv = -\iiint_{V_{[S]}}\left(\vec{E}\cdot\vec{J}^{i*} + \vec{H}^*\cdot\vec{M}^i\right)dv \quad (12.1)$$

Poynting's theorem states the conservation of energy law in EM systems.

One starts with the supposition that a given EM problem has two solutions (due to the same sources and the same boundary conditions): $\left(\vec{E}^a,\vec{H}^a\right)$ and $\left(\vec{E}^b,\vec{H}^b\right)$. The difference field is then formed:

$$\begin{vmatrix} \delta\vec{E} = \vec{E}^a - \vec{E}^b \\ \delta\vec{H} = \vec{H}^a - \vec{H}^b \end{vmatrix} \quad (12.2)$$

Since the difference field has no sources, it will satisfy the source-free form of (12.1):

$$\oiint_S \left(\delta\vec{E}\times\delta\vec{H}^*\right)d\vec{s} + j\omega\iiint_{V_{[S]}}\left(\mu\,|\,\delta\vec{H}\,|^2 - \varepsilon\,|\,\delta\vec{E}\,|^2\right)dv + \iiint_{V_{[S]}}\sigma\,|\,\delta\vec{E}\,|^2\,dv = 0 \quad (12.3)$$

Since both fields satisfy the same boundary conditions at *S*, then $\delta\vec{E} = 0$ and $\delta\vec{H} = 0$ over *S*. This leaves us with

$$j\omega\iiint_{V_{[S]}}\left(\mu\,|\,\delta\vec{H}\,|^2 - \varepsilon\,|\,\delta\vec{E}\,|^2\right)dv + \iiint_{V_{[S]}}\sigma\,|\,\delta\vec{E}\,|^2\,dv = 0, \quad (12.4)$$

which is true only if

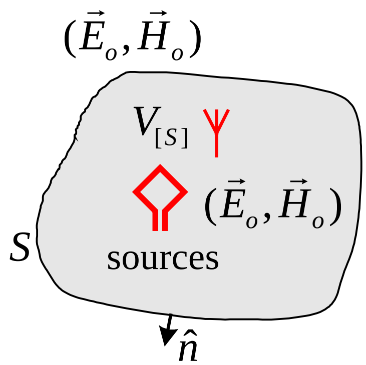
$$\begin{vmatrix} \omega\iiint_{V_{[S]}}\left(\mu\,|\,\delta\vec{H}\,|^2 - \varepsilon\,|\,\delta\vec{E}\,|^2\right)dv = 0 \\ \iiint_{V_{[S]}}\sigma\,|\,\delta\vec{E}\,|^2\,dv = 0 \end{vmatrix} \quad (12.5)$$

If we assume some dissipation, however slight, equations (12.5) are satisfied only if $\delta\vec{E} = \delta\vec{H} = 0$ everywhere in the volume $V_{[S]}$. This implies the uniqueness of the solution. If $\sigma = 0$, which is a physical impossibility, but is often used approximation, multiple solutions $(\delta\vec{E},\delta\vec{H})$ may exist in the form of self-resonant modes of the structure under consideration. In open problems, resonance is impossible in the whole region.

Notice that the uniqueness theorem holds if *either* $\delta\vec{E}=0$ *or* $\delta\vec{H}=0$ *is true on any part of the boundary*.

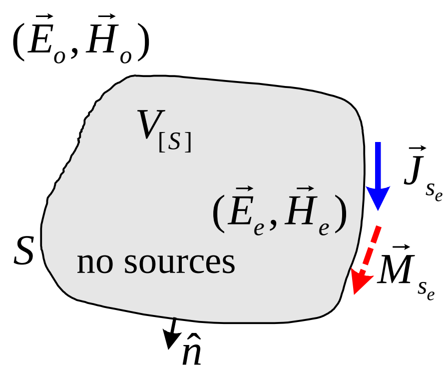
2. Equivalence principles

The equivalence principle follows from the uniqueness theorem. It allows us to build simpler to solve problems. As long as the equivalent problem preserves the boundary conditions of the original problem for the field at *S*, it is going to produce the only one possible solution for the region outside $V_{[S]}$.



(a) Original problem

$\Rightarrow$

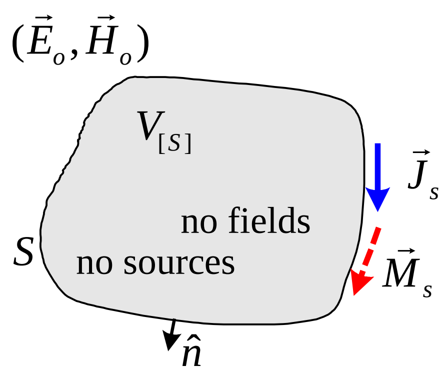


(b) General equivalent problem

$\Downarrow$

$$\begin{cases} \vec{J}_{s_e}=\hat{n}\times\left(\vec{H}_o-\vec{H}_e\right) \\ \vec{M}_{s_e}=\left(\vec{E}_o-\vec{E}_e\right)\times\hat{n} \end{cases} \quad (12.6)$$

$$\begin{cases} \vec{J}_s=\hat{n}\times\vec{H}_o \\ \vec{M}_s=\vec{E}_o\times\hat{n} \end{cases} \quad (12.7)$$



(c) Equivalent problem with zero fields

The zero-field formulation is often referred to as *Love's equivalence principle.*

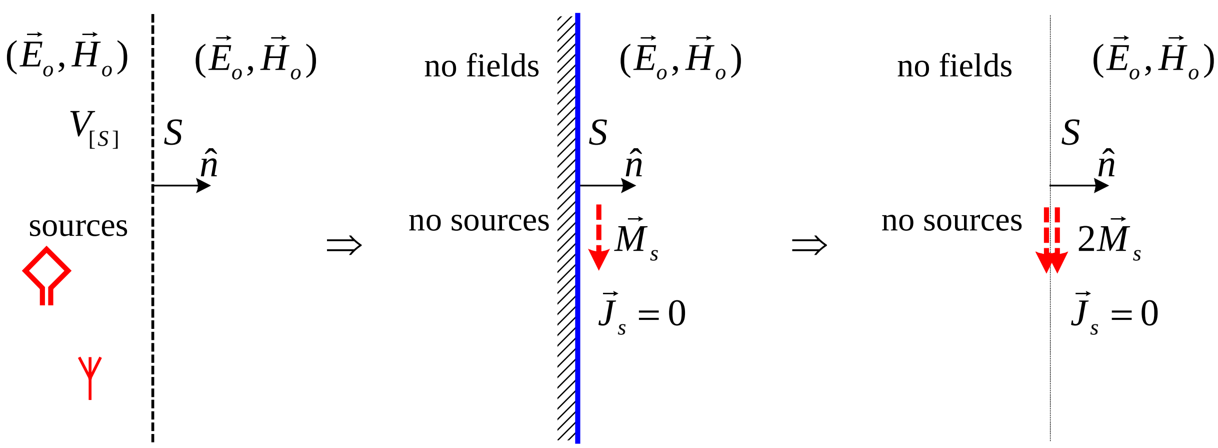
One can apply Love's equivalence principle in three different ways:

(a) One can assume that <u>the boundary *S* is a perfect conductor</u>. This eliminates the surface electric currents, i.e. $\vec{J}_s = 0$, and leaves just surface magnetic currents $\vec{M}_s$, which radiate in the presence of a perfect electric surface.

(b) One can assume that <u>the boundary *S* is a perfect magnetic conductor</u>. This eliminates the surface magnetic currents, i.e. $\vec{M}_s = 0$, and leaves just surface electric currents $\vec{J}_s$, which radiate in the presence of a perfect magnetic surface.

(c) Make no assumptions about the materials inside *S*, and define both $\vec{J}_s$ and $\vec{M}_s$ currents, which are radiating in free space (no fictitious conductors behind them). It can be shown that these equivalent currents create zero fields inside $V_{[S]}$.

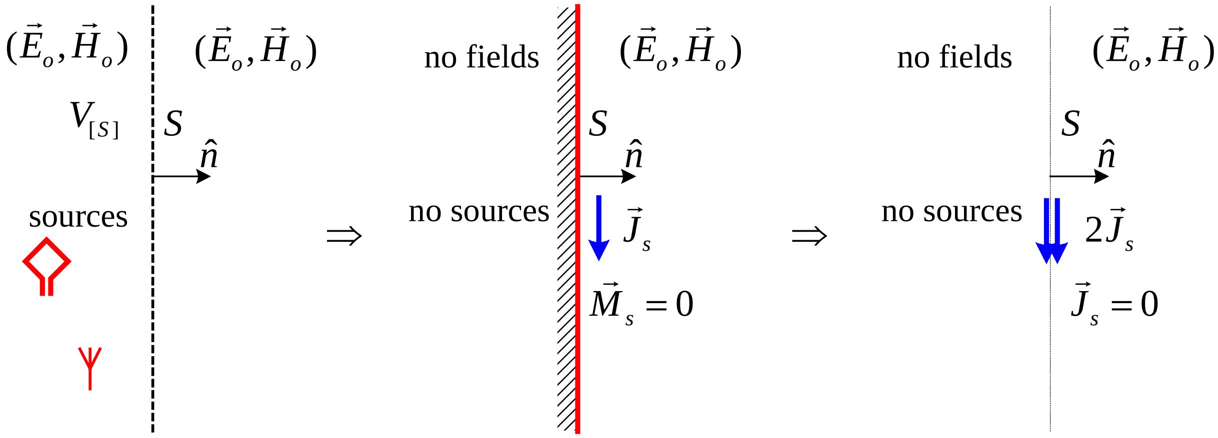
All three approaches lead to the same field solution according to the uniqueness theorem. The first two approaches are not very useful in the general case of curvilinear boundary surface *S*. However, in the case of flat infinite planes (walls), the image theory can be used to reduce the problem to an open one. Image theory can be successfully applied to curved surfaces provided the curvature's radius is large compared to the wavelength. Here is how one can implement Love's equivalence principles in conjunction with image theory.



(a) Original problem

(b) Equivalent problem - electric wall

(c) Equivalent problem - images



(a) Original problem

(b) Equivalent problem - magnetic wall

(c) Equivalent problem - images

The above approach is used to evaluate fields in half-space as excited by apertures. The field behind S is assumed known. This is enough to define equivalent surface currents. Using image theory, the open-region far-zone solutions for the vector potentials, $\vec{A}$ (resulting from $\vec{J}_s$) and $\vec{F}$ (resulting from $\vec{M}_s$), are found from:

$$\vec{A}(P) = \mu \frac{e^{-j\beta r}}{4\pi r} \iint_S \vec{J}_s(\vec{r}') e^{j\beta \hat{r}\cdot\vec{r}'} ds' \tag{12.8}$$

$$\vec{F}(P) = \varepsilon \frac{e^{-j\beta r}}{4\pi r} \iint_S \vec{M}_s(\vec{r}') e^{j\beta \hat{r}\cdot\vec{r}'} ds' \tag{12.9}$$

Here, $\hat{r}$ denotes the unit vector pointing from the origin of the coordinate system to the point of observation P. The integration point Q is specified through the radius-vector $\vec{r}'$. In the far zone, it is assumed that the field propagates radially away from the antenna. It is convenient to introduce the so-called *propagation vector*:

$$\vec{\beta} = \beta \hat{r}, \tag{12.10}$$

which characterizes both the phase constant and the direction of propagation of the wave. The vector potentials can then be written as:

$$\vec{A}(P) = \mu \frac{e^{-j\beta r}}{4\pi r} \iint_S \vec{J}_s(\vec{r}') e^{j\vec{\beta}\cdot\vec{r}'} ds' \tag{12.11}$$

$$\vec{F}(P) = \varepsilon \frac{e^{-j\beta r}}{4\pi r} \iint_S \vec{M}_s(\vec{r}') e^{j\vec{\beta}\cdot\vec{r}'} ds' \quad (12.12)$$

The relations between the far-zone fields and the vector potentials are rather simple.

$$\vec{E}_A^{far} = -j\omega(A_\theta \hat{\theta} + A_\varphi \hat{\varphi}) \quad (12.13)$$

$$\vec{H}_F^{far} = -j\omega(F_\theta \hat{\theta} + F_\varphi \hat{\varphi}) \quad (12.14)$$

Since

$$\vec{E}_F^{far} = \eta \vec{H}_F^{far} \times \vec{r}, \quad (12.15)$$

the total far-zone electric field is found as:

$$\vec{E}^{far} = \vec{E}_A^{far} + \vec{E}_F^{far} = -j\omega\left[\left(A_\theta - \eta F_\varphi\right)\hat{\theta} + \left(A_\varphi + \eta F_\theta\right)\hat{\varphi}\right] \quad (12.16)$$

Equation (12.16) involves both vector potentials as arising from both types of surface currents. Computations are reduced in half if image theory is used in conjunction with an electric or magnetic wall assumption.

3. Application of the equivalence principle to aperture problems

The equivalence principle is widely used in the analysis of aperture antennas. To calculate exactly the far fields, the exact field distribution at the aperture is needed. In the case of exact knowledge of the aperture field distribution, all three approaches given above will produce the same results. However, such exact knowledge of the aperture field distribution is usually impossible, and certain approximations are used. Then, the three equivalence-principle approaches produce slightly different results, the consistency being dependent on how accurate our knowledge about the aperture field is. Usually, it is assumed that the field is to be determined in half-space, leaving the feed and the antenna behind a infinite wall S (electric or magnetic). The aperture of the antenna S_A is this portion of S where we have an approximate knowledge of the field distribution based on the type of the feed line or the incident wave illuminating the aperture. This is the so-called *physical optics* approximation, which certainly is more accurate than the *geometrical optics* approach of ray tracing. The larger the aperture (as compared to the wavelength), the more accurate the approximation based on the incident wave.

Let us assume that the fields at the aperture are known: $\vec{E}_a, \vec{H}_a$, and they are zero everywhere else at S. The equivalent current densities are:

$$\left| \begin{aligned} \vec{J}_s &= \hat{n} \times \vec{H}_a \\ \vec{M}_s &= \vec{E}_a \times \hat{n} \end{aligned} \right. \tag{12.17}$$

Using (12.17) in (12.11) and (12.12) produces:

$$\vec{A}(P) = \mu \frac{e^{-j\beta r}}{4\pi r} \hat{n} \times \iint_S \vec{H}_a e^{j\vec{\beta}\cdot\vec{r}'} ds' \tag{12.18}$$

$$\vec{F}(P) = -\varepsilon \frac{e^{-j\beta r}}{4\pi r} \hat{n} \times \iint_S \vec{E}_a e^{j\vec{\beta}\cdot\vec{r}'} ds' \tag{12.19}$$

The radiation integrals in (12.18) and (12.19) will be denoted shortly as:

$$\vec{\mathcal{I}}^H = \iint_S \vec{H}_a e^{j\vec{\beta}\cdot\vec{r}'} ds' \tag{12.20}$$

$$\vec{\mathcal{I}}^E = \iint_S \vec{E}_a e^{j\vec{\beta}\cdot\vec{r}'} ds' \tag{12.21}$$

One can find general vector expression for the far-field $\vec{E}$ vector making use of equation (12.16) written as:

$$\vec{E}^{far} = -j\omega\vec{A} - j\omega\eta\vec{F} \times \hat{r}, \tag{12.22}$$

where ***the longitudinal*** A_r ***component is to be neglected***. Substituting (12.18) and (12.19) yields:

$$\vec{E}^{far} = -j\beta \frac{e^{-j\beta r}}{4\pi r} \hat{r} \times \iint_{S_A} \left[\hat{n} \times \vec{E}_a - \eta \hat{r} \times \left(\hat{n} \times \vec{H}_a \right) \right] e^{j\vec{\beta}\cdot\vec{r}'} ds' \tag{12.23}$$

This is the full vector form of the radiated field resulting from the aperture field, and it is referred to as the *vector diffraction integral* (or *vector Kirchhoff integral*).

We shall now consider a practical case of a flat aperture lying in the $x-y$ plane with $\hat{n} \equiv \hat{z}$. Then:

$$\vec{A} = \mu \frac{e^{-j\beta r}}{4\pi r}\left(-\mathcal{J}_y^H \hat{x} + \mathcal{J}_x^H \hat{y}\right) \tag{12.24}$$

$$\vec{F} = -\varepsilon \frac{e^{-j\beta r}}{4\pi r}\left(-\mathcal{J}_y^E \hat{x} + \mathcal{J}_x^E \hat{y}\right) \tag{12.25}$$

The integrals in the above expressions can be explicitly written for this case in which $\vec{r}' = x'\hat{x} + y'\hat{y}$:

$$\mathcal{J}_x^E = \iint_{S_A} E_{a_x}(x', y') e^{j\beta(x'\sin\theta\cos\varphi + y'\sin\theta\sin\varphi)} dx'dy' \tag{12.26}$$

$$\mathcal{J}_y^E = \iint_{S_A} E_{a_y}(x', y') e^{j\beta(x'\sin\theta\cos\varphi + y'\sin\theta\sin\varphi)} dx'dy' \tag{12.27}$$

$$\mathcal{J}_x^H = \iint_{S_A} H_{a_x}(x', y') e^{j\beta(x'\sin\theta\cos\varphi + y'\sin\theta\sin\varphi)} dx'dy' \tag{12.28}$$

$$\mathcal{J}_y^H = \iint_{S_A} H_{a_y}(x', y') e^{j\beta(x'\sin\theta\cos\varphi + y'\sin\theta\sin\varphi)} dx'dy' \tag{12.29}$$

Note that the above integrals are exactly the double inverse Fourier transforms of the aperture field's components.

The vector potentials in spherical terms are:

$$\vec{A} = \mu \frac{e^{-j\beta r}}{4\pi r}\left[\hat{\theta}\cos\theta\left(\mathcal{J}_x^H \sin\varphi - \mathcal{J}_y^H \cos\varphi\right) + \hat{\varphi}\left(\mathcal{J}_x^H \cos\varphi + \mathcal{J}_y^H \sin\varphi\right)\right] \tag{12.30}$$

$$\vec{F} = -\varepsilon \frac{e^{-j\beta r}}{4\pi r}\left[\hat{\theta}\cos\theta\left(\mathcal{J}_x^E \sin\varphi - \mathcal{J}_y^E \cos\varphi\right) + \hat{\varphi}\left(\mathcal{J}_x^E \cos\varphi + \mathcal{J}_y^E \sin\varphi\right)\right] \tag{12.31}$$

By substituting the above expressions in (12.16), one obtains the far $\vec{E}$ field components as:

$$E_\theta = j\beta \frac{e^{-j\beta r}}{4\pi r}[\mathcal{J}_x^E \cos\varphi + \mathcal{J}_y^E \sin\varphi + \eta\cos\theta(\mathcal{J}_y^H \cos\varphi - \mathcal{J}_x^H \sin\varphi)] \tag{12.32}$$

$$E_\varphi = j\beta \frac{e^{-j\beta r}}{4\pi r}[-\eta(\mathcal{J}_x^H \cos\varphi + \mathcal{J}_y^H \sin\varphi) + \cos\theta(\mathcal{J}_y^E \cos\varphi - \mathcal{J}_x^E \sin\varphi)] \tag{12.33}$$

For apertures mounted on a conducting plane, the preferred equivalent model is the one with electric wall with magnetic current density

$$\vec{M}_s = 2 \cdot \left(\vec{E}_a \times \hat{n} \right) \tag{12.34}$$

radiating in open space. The solution, of course, is valid only for $z \geq 0$. In this case, $\vec{\mathcal{J}}^H = 0$.

For apertures in open space, the dual current formulation is used. Then, a usual assumption is that the aperture fields are related as in the TEM-wave case:

$$\vec{H}_a = \frac{1}{\eta} \hat{z} \times \vec{E}_a \tag{12.35}$$

This implies that

$$\vec{\mathcal{J}}^H = \frac{1}{\eta} \hat{z} \times \vec{\mathcal{J}}^E \quad \text{or} \quad \mathcal{J}_x^H = -\frac{\mathcal{J}_y^E}{\eta}, \mathcal{J}_y^H = \frac{\mathcal{J}_x^E}{\eta} \tag{12.36}$$

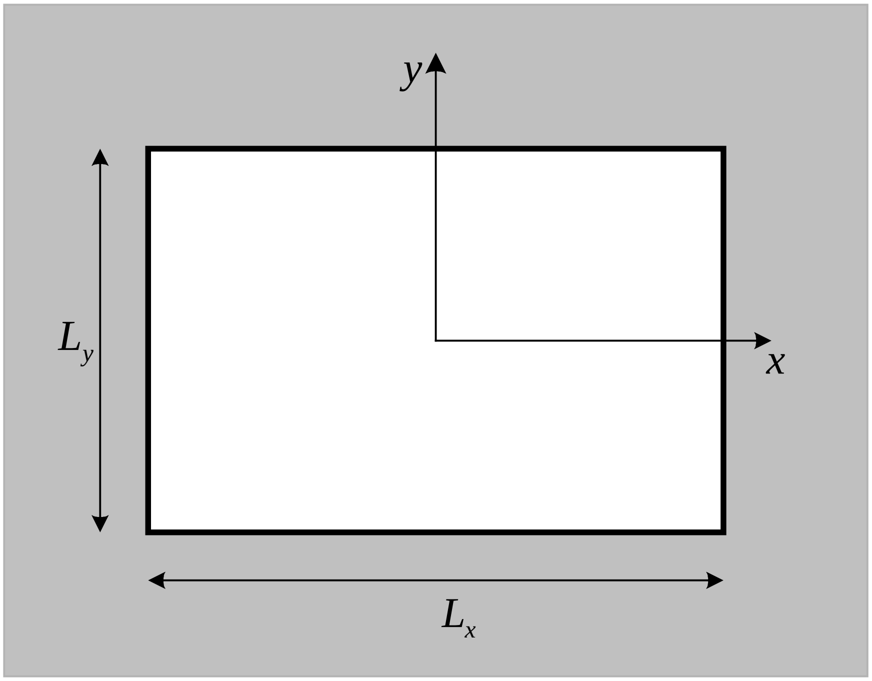
This assumption is valid for moderate and high-gain apertures; therefore, the apertures should be at least a couple of wavelengths in extent. The above assumptions reduce (12.32)-(12.33) to:

$$E_\theta = j\beta\eta \frac{e^{-j\beta r}}{4\pi r} \frac{(1+\cos\theta)}{2} \left[\mathcal{J}_x^E \cos\varphi + \mathcal{J}_y^E \sin\varphi \right] \tag{12.37}$$

$$E_\varphi = j\beta\eta \frac{e^{-j\beta r}}{4\pi r} \frac{(1+\cos\theta)}{2} \left[\mathcal{J}_y^E \cos\varphi - \mathcal{J}_x^E \sin\varphi \right] \tag{12.38}$$

4. The uniform rectangular aperture on an infinite ground plane

A rectangular aperture is defined in the $x-y$ plane as shown below.



If the fields are uniform in amplitude and phase across the aperture, it is referred to as a *uniform rectangular aperture*. Let us assume that the aperture field is *y*-polarized.

$$\vec{E}_a = E_0\hat{y}, \quad |x| \le \frac{L_x}{2} \text{ and } |y| \le \frac{L_y}{2} \tag{12.39}$$

According to the equivalence principle, we assume an electric wall at $z = 0$, where the equivalent magnetic current density is given by $\vec{M}_{s_e} = \vec{E} \times \hat{n}$. Applying image theory, one can find the equivalent sources radiating in open space as:

$$\vec{M}_s = 2\vec{M}_{s_e} = 2E_0\hat{y} \times \hat{z} = 2E_0\hat{x} \tag{12.40}$$

The only non-zero radiation integral is:

$$\mathcal{J}_y^E = 2E_0 \int_{-L_x/2}^{L_x/2} e^{j\beta x' \sin\theta\cos\varphi} dx' \cdot \int_{-L_y/2}^{L_y/2} e^{j\beta y' \sin\theta\sin\varphi} dy' =$$

$$= 2E_0 L_x L_y \frac{\sin\left[\frac{\beta L_x}{2}\sin\theta\cos\varphi\right]}{\frac{\beta L_x}{2}\sin\theta\cos\varphi} \cdot \frac{\sin\left[\frac{\beta L_y}{2}\sin\theta\sin\varphi\right]}{\frac{\beta L_y}{2}\sin\theta\sin\varphi} \quad (12.41)$$

It is appropriate to introduce the pattern variables:

$$\left| \begin{array}{l} u = \frac{\beta L_x}{2}\sin\theta\cos\varphi \\ v = \frac{\beta L_y}{2}\sin\theta\sin\varphi \end{array} \right. \quad (12.42)$$

The complete radiation fields are found by substituting (12.41) in (12.32) and (12.33):

$$\left| \begin{array}{l} E_\theta = j\beta \frac{e^{-j\beta r}}{2\pi r} E_0 L_x L_y \sin\varphi \frac{\sin u}{u} \frac{\sin v}{v} \\ E_\varphi = j\beta \frac{e^{-j\beta r}}{2\pi r} E_0 L_x L_y \cos\theta\cos\varphi \frac{\sin u}{u} \frac{\sin v}{v} \end{array} \right. \quad (12.43)$$

The total-field amplitude pattern is, therefore:

$$|\bar{E}| = F(\theta,\varphi) = \sqrt{\sin^2\varphi + \cos^2\theta\cos^2\varphi} \cdot \frac{\sin u}{u} \frac{\sin v}{v} =$$

$$= \sqrt{1 - \sin^2\theta\cos^2\varphi} \cdot \frac{\sin u}{u} \frac{\sin v}{v} \quad (12.44)$$

The principal plane patterns are:

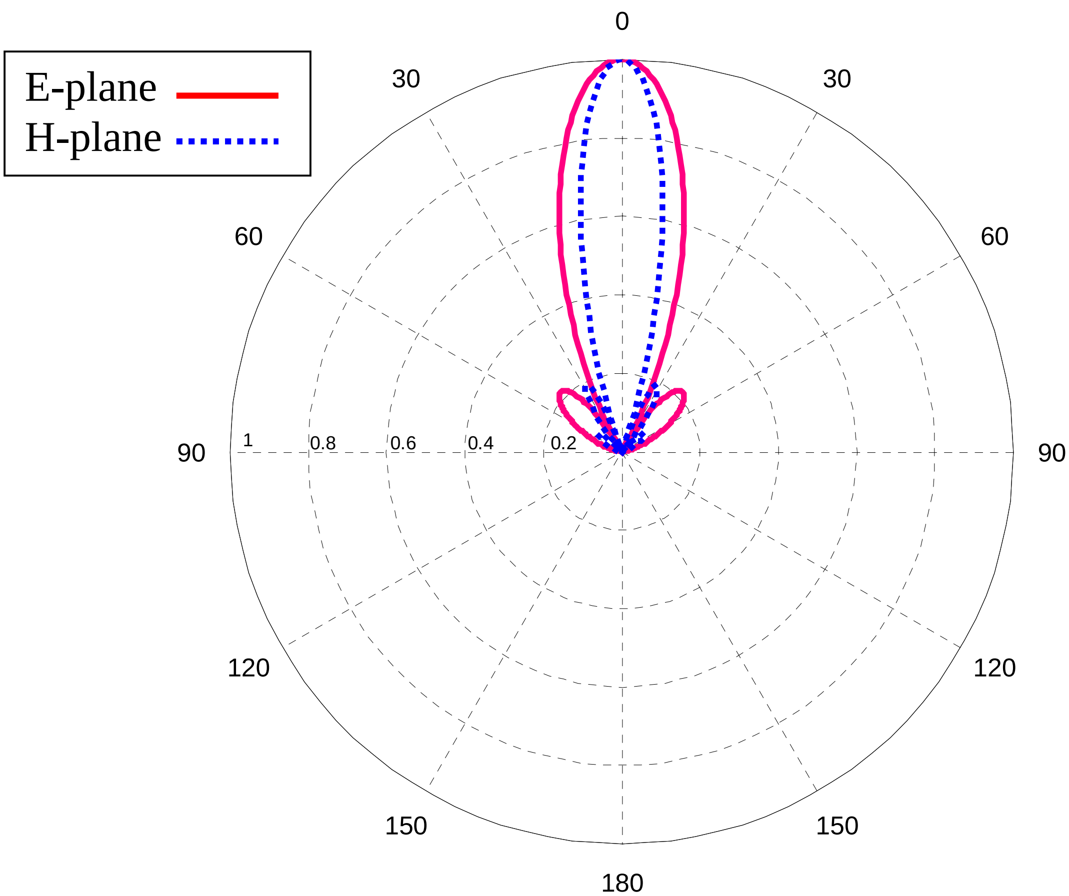
<u>*E*-plane pattern</u> $(\varphi = \pi / 2)$

$$\bar{E}_\theta = \frac{\sin\left(\dfrac{\beta L_y}{2}\sin\theta\right)}{\left(\dfrac{\beta L_y}{2}\sin\theta\right)} \tag{12.45}$$

<u>*H*-plane pattern</u> $(\varphi = 0)$

$$\bar{E}_\varphi = \cos\theta \frac{\sin\left(\dfrac{\beta L_x}{2}\sin\theta\right)}{\left(\dfrac{\beta L_x}{2}\sin\theta\right)} \tag{12.46}$$

Principle patterns for aperture of size: $L_x = 3\lambda$, $L_y = 2\lambda$

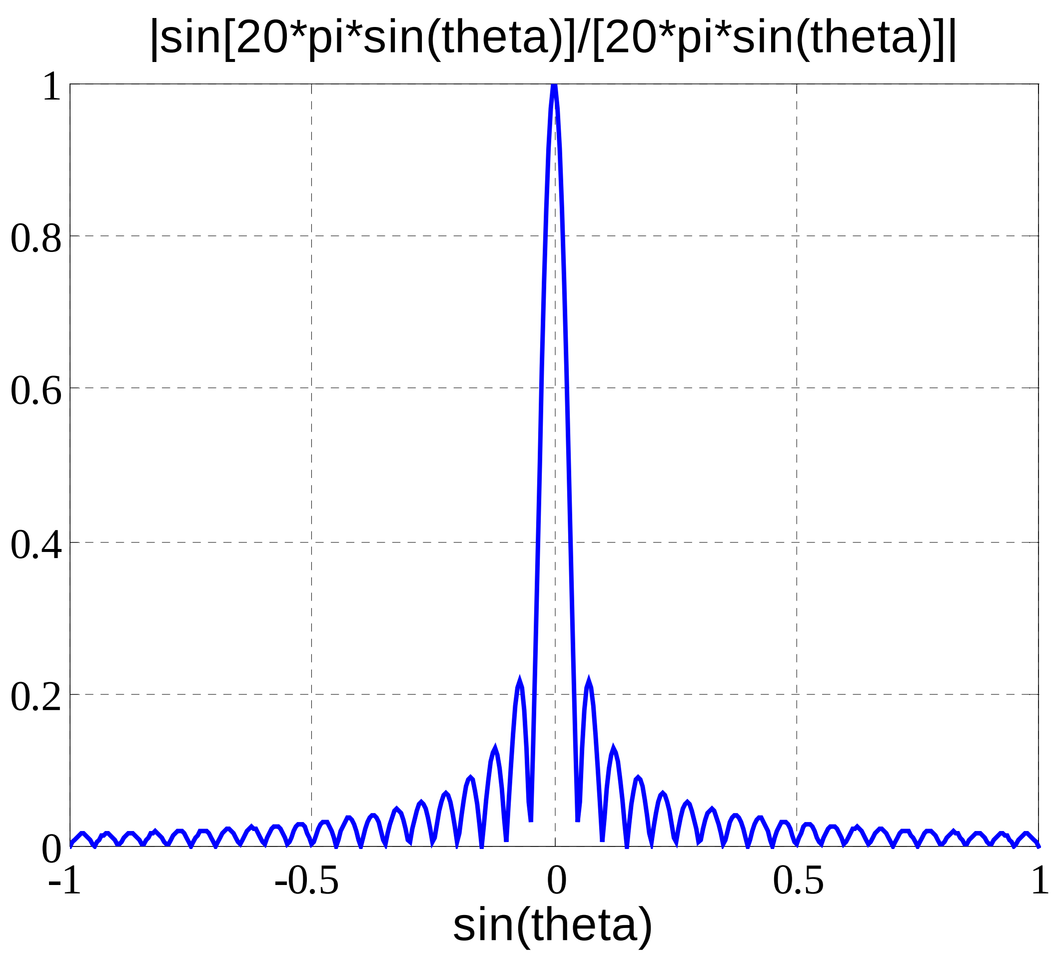


For electrically large apertures, the main beam is narrow and the $\sqrt{1-\sin^2\theta\cos^2\varphi}$ in (12.44) is negligible, i.e. it is roughly equal to 1 for all observation angles within the main beam. That is why, in the theory of large arrays, it is assumed that the amplitude pattern of a rectangular aperture is:

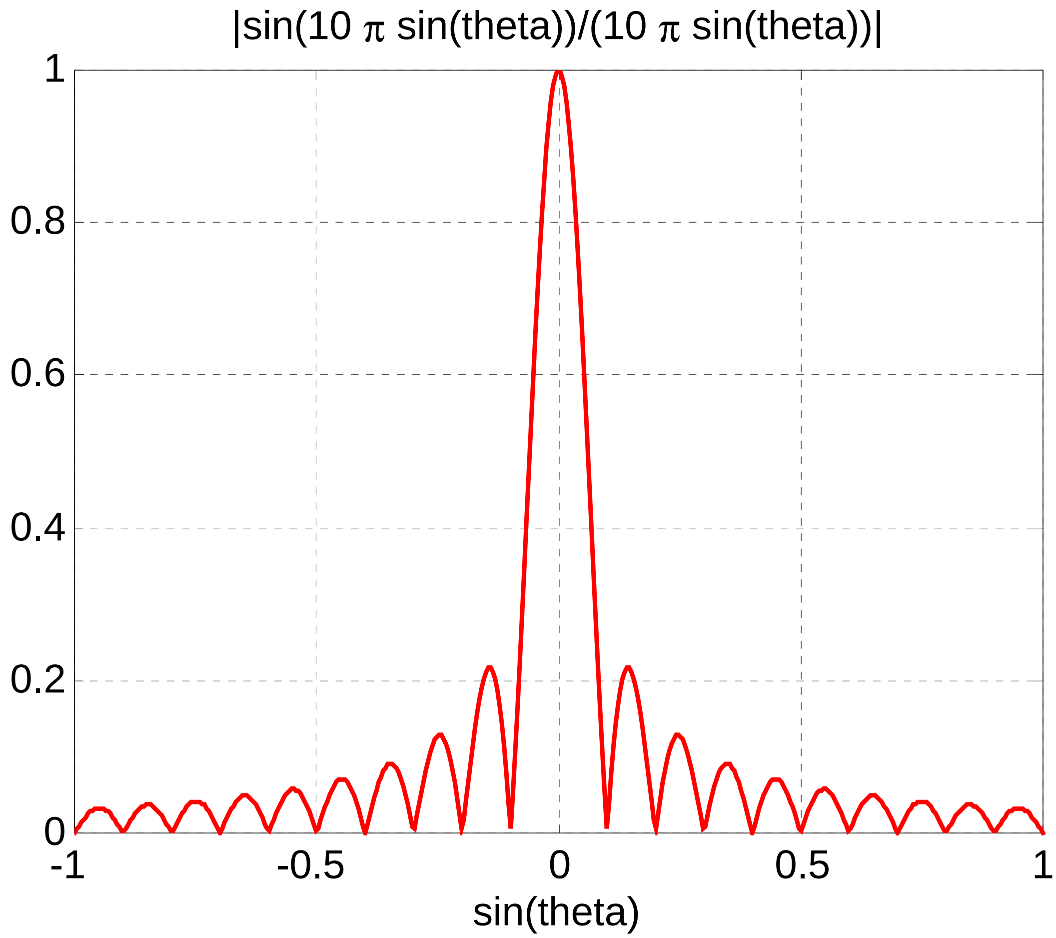
$$f(u,v) \simeq \left| \frac{\sin u}{u} \frac{\sin v}{v} \right| \qquad (12.47)$$

where $u = \frac{\beta L_x}{2} \sin\theta \cos\varphi$ and $v = \frac{\beta L_y}{2} \sin\theta \sin\varphi$.

Here is a view of the $|\sin u / u|$ function for $L_x = 20\lambda$ and $\varphi = 0^\circ$ (H-plane pattern):



Here is a view of the $|\sin v / v|$ function for $L_y = 10\lambda$ and $\varphi = 90^\circ$ (E-plane pattern):



<u>Beamwidths</u>

(a) first-null beamwidth

One needs the location of the first nulls in the pattern in order to calculate the FNBW. The nulls of the E-plane pattern are determined from (12.45) as:

$$\frac{\beta L_y}{2} \sin\theta_{/\theta=\theta_n} = n\pi, \quad n = 1, 2, \dots \tag{12.48}$$

$$\Rightarrow \theta_n = \arcsin\left(\frac{n\lambda}{L_y}\right), \text{ rad} \tag{12.49}$$

The first null occurs at $n = 1$.

$$\Rightarrow FNBW_E = 2\theta_n = 2\arcsin\left(\frac{\lambda}{L_y}\right), \text{ rad} \tag{12.50}$$

In a similar fashion, $FNBW_H$ is determined to be:

$$FNBW_H = 2\arcsin\left(\frac{\lambda}{L_x}\right), \text{ rad} \tag{12.51}$$

(b) half-power beamwidth

The half-power point in the E-plane occurs when

$$\frac{\sin\left(\frac{\beta L_y}{2}\sin\theta\right)}{\left(\frac{\beta L_y}{2}\sin\theta\right)} = \frac{1}{\sqrt{2}} \tag{12.52}$$

or

$$\frac{\beta L_y}{2}\sin\theta_{/\theta=\theta_h} = 1.391 \tag{12.53}$$

$$\Rightarrow \theta_h = \arcsin\left(\frac{0.443\lambda}{L_y}\right), \text{ rad} \tag{12.54}$$

$$HPBW_E = 2\arcsin\left(\frac{0.443\lambda}{L_y}\right) \tag{12.55}$$

A first-order approximation is possible for very small arguments in (12.55), i.e. when $L_y \gg 0.443\lambda$ (large aperture):

$$HPBW_E \simeq 0.886\frac{\lambda}{L_y} \tag{12.56}$$

The half-power beamwidth in the H-plane is analogous:

$$HPBW_H = 2\arcsin\left(\frac{0.443\lambda}{L_x}\right) \tag{12.57}$$

Side-lobe level

It is obvious from the properties of the $|\sin x / x|$ function that the first side lobe has the largest maximum of all side lobes, and it is:

$$|E_\theta(\theta=\theta_s)| = \left|\frac{\sin 4.494}{4.494}\right| = 0.217 = -13.26\text{, dB} \qquad (12.58)$$

When evaluating side-lobe levels and beamwidths in the H-plane, one has to include the $\cos\theta$ factor, too. The smaller the aperture, the less important this factor is.

Directivity

In a general approach to the calculation of the directivity, the total radiated power Π has to be calculated first using the far-field pattern expression (12.44).

$$D_0 = \frac{4\pi}{\Omega_A} = 4\pi\frac{U_{\max}}{\Pi_{rad}} \qquad (12.59)$$

Here,

$$U(\theta,\varphi) = \frac{1}{2\eta}\left[|E_\theta|^2 + |E_\varphi|^2\right]r^2 = U_{\max}|F(\theta,\varphi)|^2 \qquad (12.60)$$

$$\Omega_A = \int_0^{2\pi}\int_0^{\pi}|F(\theta,\varphi)|^2 \sin\theta d\theta d\varphi \qquad (12.61)$$

However, in the case of an aperture illuminated by a TEM wave, one can use a simpler approach. Generally, for all aperture antennas, the assumption of a uniform TEM wave at the aperture ($\vec{E} = \hat{y}E_0$),

$$\vec{H}_a = -\hat{x}\frac{E_0}{\eta}, \qquad (12.62)$$

is quite accurate (although η is not necessarily the intrinsic impedance of vacuum). The far-field components in this case were already derived in (12.37) and (12.38). They lead to the following expression for the radiation intensity:

$$U(\theta,\varphi) = \frac{\beta^2}{32\pi^2\eta}(1+\cos\theta)^2\left[|\mathcal{J}_x^E|^2 + |\mathcal{J}_y^E|^2\right] \qquad (12.63)$$

The maximum value of the function in (12.63) is easily derived after substituting the radiation integrals from (12.26) and (12.27):

$$U_{\max} = \frac{\beta^2}{8\pi^2\eta}\left|\iint_{S_A} \vec{E}_a ds'\right|^2 \qquad (12.64)$$

The integration of the radiation intensity (12.63) over a closed sphere is in general not easy. It can be avoided by observing that the total power reaching the far zone must have passed through the aperture in the first place. In the general aperture case, this power is determined as:

$$\Pi_{rad} = \oiint_S \vec{P}_{av} \cdot d\vec{s} = \frac{1}{2\eta}\iint_{S_A} |\vec{E}_a|^2\, ds \qquad (12.65)$$

Substituting (12.64) and (12.65) in (12.59) finally yields:

$$D_0 = \frac{4\pi}{\lambda^2}\frac{\left|\iint_{S_A} \vec{E}_a ds'\right|^2}{\iint_{S_A} |\vec{E}_a|^2\, ds'} \qquad (12.66)$$

In the case of a uniform rectangular aperture,

$$\Pi = L_x L_y \frac{|E_0|^2}{2\eta} \qquad (12.67)$$

$$U_{\max} = \left(\frac{L_x L_y}{\lambda}\right)^2 \frac{|E_0|^2}{2\eta} \qquad (12.68)$$

Thus, the directivity is found to be:

$$D_0 = 4\pi\frac{U_{\max}}{\Pi} = \frac{4\pi}{\lambda^2} L_x L_y = \frac{4\pi}{\lambda^2} A_p = \frac{4\pi}{\lambda^2} A_{eff} \qquad (12.69)$$

The physical and the effective areas of a uniform aperture are equal.

5. The uniform rectangular aperture in open space

Now, we shall examine the same aperture when it is ***not*** mounted on a ground plane. The field distribution is the same as in (12.39) but now the $\vec{H}$ field must be defined, too, in order to apply the general form of the equivalence principle with both types of surface currents.

$$\left.\begin{aligned} \vec{E}_a &= \hat{y}E_0 \\ \vec{H}_a &= -\hat{x}\frac{E_0}{\eta} \end{aligned}\right\}, \quad \begin{aligned} -L_x/2 \le x' \le L_x/2 \\ -L_y/2 \le y' \le L_y/2 \end{aligned} \qquad (12.70)$$

Above, again an assumption was made that there is a direct relation between the electric and the magnetic field components.

To form the equivalent problem, an infinite surface is chosen again to extend in the $z = 0$ plane. Over the entire surface, the equivalent $\vec{J}_s$ and $\vec{M}_s$ must be defined. Both $\vec{J}_s$ and $\vec{M}_s$ are not zero outside the aperture in the $z = 0$ plane because the field is not zero there. Moreover, the field is *not known a priori* outside the aperture. Thus, the exact equivalent problem cannot be built in practice (at least, not by making use of the infinite plane model).

The usual assumption made is that $\vec{E}_a$ and $\vec{H}_a$ are zero outside the aperture in the $z = 0$ plane, and, therefore, so are the equivalent currents $\vec{J}_s$ and $\vec{M}_s$:

$$\left.\begin{aligned} \vec{M}_s &= -\hat{n}\times\vec{E}_a = \underbrace{-\hat{z}\times\hat{y}}_{\hat{x}}E_0 \\ \vec{J}_s &= \hat{n}\times\vec{H}_a = \underbrace{\hat{z}\times(-\hat{x})}_{-\hat{y}}\frac{E_0}{\eta} \end{aligned}\right\}, \quad \begin{aligned} -L_x/2 \le x' \le L_x/2 \\ -L_y/2 \le y' \le L_y/2 \end{aligned} \qquad (12.71)$$

Since the equivalent currents are related via the TEM-wave assumption, only the integral $\mathcal{J}_y^E$ is needed for substitution in the far field expressions derived in (12.37) and (12.38).

$$\mathcal{J}_y^E = 2E_0 \int_{-L_x/2}^{L_x/2} e^{j\beta x' \sin\theta \cos\varphi} dx' \cdot \int_{-L_y/2}^{L_y/2} e^{j\beta y' \sin\theta \sin\varphi} dy' =$$

$$= 2E_0 L_x L_y \frac{\sin\left[\frac{\beta L_x}{2}\sin\theta\cos\varphi\right]}{\frac{\beta L_x}{2}\sin\theta\cos\varphi} \cdot \frac{\sin\left[\frac{\beta L_y}{2}\sin\theta\sin\varphi\right]}{\frac{\beta L_y}{2}\sin\theta\sin\varphi} \quad (12.72)$$

Now, the far-field components are obtained by substituting in (12.37) and (12.38):

$$E_\theta = C\sin\varphi \frac{(1+\cos\theta)}{2}\frac{\sin u}{u}\frac{\sin v}{v}$$
$$E_\varphi = C\cos\varphi \frac{(1+\cos\theta)}{2}\frac{\sin u}{u}\frac{\sin v}{v} \quad (12.73)$$

where:

$$C = j\beta L_x L_y E_0 \frac{e^{-j\beta r}}{2\pi r};$$
$$u = \frac{\beta L_x}{2}\sin\theta\cos\varphi;$$
$$v = \frac{\beta L_y}{2}\sin\theta\sin\varphi.$$

The far-field expressions in (12.73) would be identical to those of the aperture mounted on a ground plane if $\cos\theta$ were replaced by 1. Thus, for small values of θ, the patterns of both apertures are practically identical.

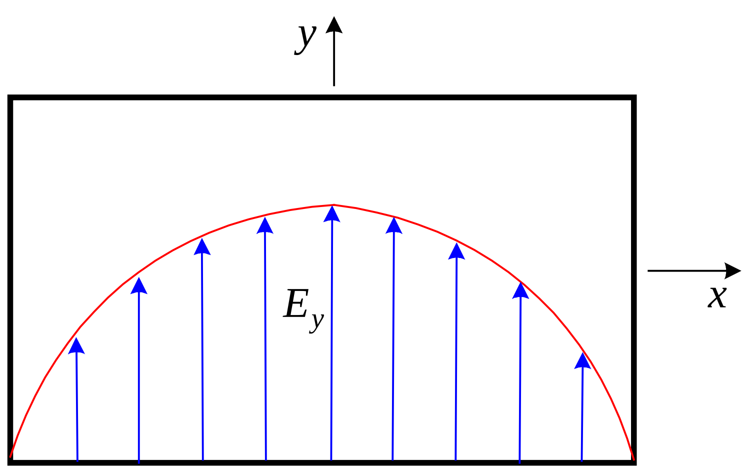
An exact analytical evaluation of the directivity is difficult. However, according to the approximations made, the directivity formula derived in (12.66) should provide accurate enough value. According to (12.66), the directivity is the same as in the case of the aperture mounted on a ground plane.

6. The tapered rectangular aperture on a ground plane

The uniform rectangular aperture has the maximum possible effective area (for an aperture-type antenna) equal to its physical area. This also implies that it has the highest possible directivity for all constant-phase excitations of a rectangular aperture. However, directivity is not the only important factor in the design of an antenna. A factor that frequently comes into a conflict with the directivity is the side-lobe level (SLL). The uniform distribution excitation produces the highest SLL of all constant-phase excitations of a rectangular aperture. It will be shown that the reduction of SLL can be achieved by tapering the equivalent sources distribution from a maximum at the aperture's center to zero values at its edges.

One very practical aperture of tapered source distribution is the open rectangular waveguide. The dominant (TE_{10}) mode has the following distribution:

$$\vec{E}_a = \hat{y} E_0 \cos\left(\frac{\pi}{L_x} x'\right), \quad \begin{cases} -L_x/2 \le x' \le L_x/2 \\ -L_y/2 \le y' \le L_y/2 \end{cases} \tag{12.74}$$



The general procedure for the far-field analysis is the same as before (in Section 4). The only difference is in the field distribution. Again, only the integral $\mathcal{J}_y^E$ is to be evaluated.

$$\mathcal{J}_y^E = 2E_0 \int_{-L_x/2}^{L_x/2} \cos\left(\frac{\pi}{L_x} x'\right) e^{j\beta x' \sin\theta\cos\varphi} dx' \cdot \int_{-L_y/2}^{L_y/2} e^{j\beta y' \sin\theta\sin\varphi} dy' \quad (12.75)$$

The integral of the y variable was already encountered in (12.41):

$$I(y) = \int_{-L_y/2}^{L_y/2} e^{j\beta y' \sin\theta\sin\varphi} dy' = L_y \frac{\sin\left[\frac{\beta L_y}{2}\sin\theta\sin\varphi\right]}{\frac{\beta L_y}{2}\sin\theta\sin\varphi} \qquad (12.76)$$

The integral of the x variable is easily solved:

$$I(x) = \int_{-L_x/2}^{L_x/2} \cos\left(\frac{\pi}{L_x} x'\right) e^{j\beta x' \sin\theta\cos\varphi} dx' =$$

$$= \int_{-L_x/2}^{L_x/2} \cos\left(\frac{\pi}{L_x} x'\right) \left[\cos\left(\beta x' \sin\theta\cos\varphi\right) + j\sin\left(\beta x' \sin\theta\cos\varphi\right)\right] dx' =$$

$$= \frac{1}{2}\int_{-L_x/2}^{L_x/2} \left\{\cos\left[\left(\frac{\pi}{L_x} - \beta\sin\theta\cos\varphi\right)x'\right] + \cos\left[\left(\frac{\pi}{L_x} + \beta\sin\theta\cos\varphi\right)x'\right]\right\} dx' +$$

$$+ \frac{j}{2}\int_{-L_x/2}^{L_x/2} \left\{\sin\left[\left(\beta\sin\theta\cos\varphi - \frac{\pi}{L_x}\right)x'\right] + \cos\left[\left(\beta\sin\theta\cos\varphi + \frac{\pi}{L_x}\right)x'\right]\right\} dx'$$

$$\Rightarrow I(x) = \frac{\pi L_x}{2} \frac{\cos\left(\frac{\beta L_x}{2}\sin\theta\cos\varphi\right)}{\left(\frac{\pi}{2}\right)^2 - \frac{\beta L_x}{2}\sin\theta\cos\varphi}$$

$$\Rightarrow \mathcal{J}_y^E = \pi E_0 L_x L_y \frac{\cos\left(\frac{\beta L_x}{2}\sin\theta\cos\varphi\right)}{\left[\left(\frac{\pi}{2}\right)^2 - \underbrace{\frac{\beta L_x}{2}\sin\theta\cos\varphi}_{u}\right]} \frac{\sin\left[\frac{\beta L_y}{2}\sin\theta\sin\varphi\right]}{\underbrace{\left(\frac{\beta L_y}{2}\sin\theta\sin\varphi\right)}_{v}} \quad (12.77)$$

To derive the far-field components, (12.77) is substituted in (12.32) and (12.33).

$$E_\theta = -\frac{\pi}{2} C \sin\varphi \frac{\cos u}{\left[u^2 - \left(\frac{\pi}{2}\right)^2\right]} \frac{\sin v}{v}$$

$$E_\varphi = -\frac{\pi}{2} C \cos\theta \cos\varphi \frac{\cos u}{\left[u^2 - \left(\frac{\pi}{2}\right)^2\right]} \frac{\sin v}{v} \tag{12.78}$$

where:

$$C = j\beta L_x L_y E_0 \frac{e^{-j\beta r}}{2\pi r};$$

$$u = \frac{\beta L_x}{2} \sin\theta \cos\varphi;$$

$$v = \frac{\beta L_y}{2} \sin\theta \sin\varphi.$$

Principle plane patterns

In the E-plane, the aperture is not tapered. As expected, the E-plane principal pattern is the same as that of a uniform aperture.

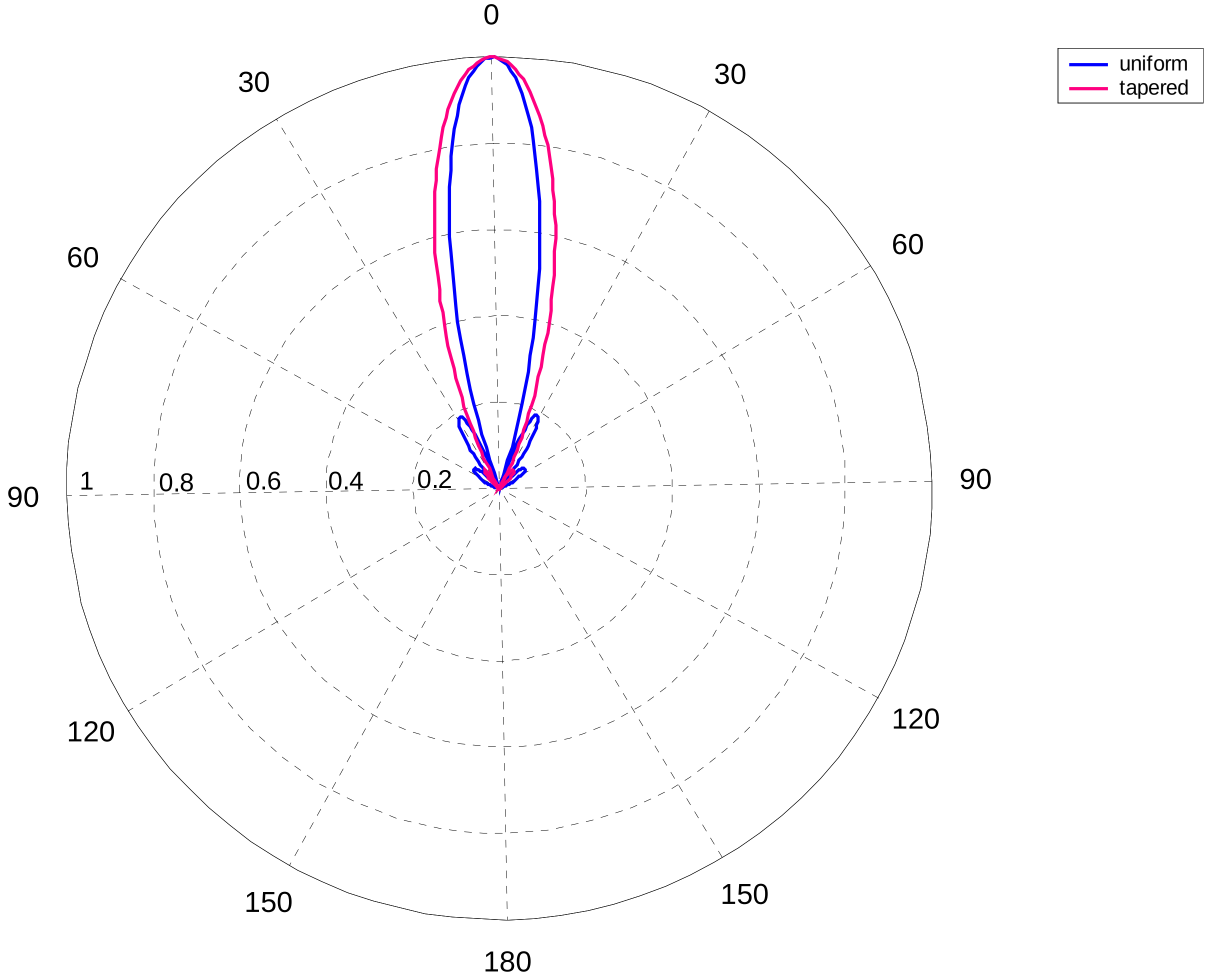
E-plane ($\varphi = 90^\circ$):

$$\overline{E}_\theta = \frac{\sin\left(\frac{\beta L_y}{2}\sin\theta\right)}{\left(\frac{\beta L_y}{2}\sin\theta\right)} \tag{12.79}$$

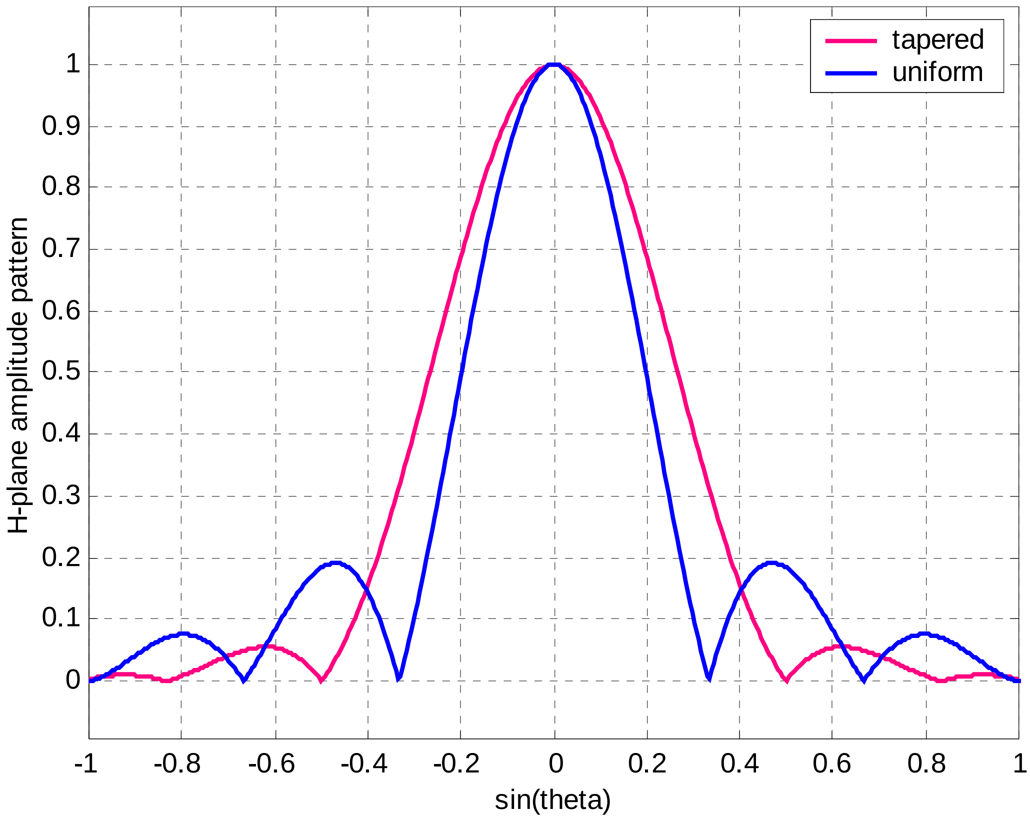
H-plane ($\varphi = 0^\circ$):

$$\overline{E}_\varphi = \cos\theta \frac{\cos\left(\frac{\beta L_x}{2}\sin\theta\right)}{\left(\frac{\beta L_x}{2}\sin\theta\right)^2 - \left(\frac{\pi}{2}\right)^2} \tag{12.80}$$

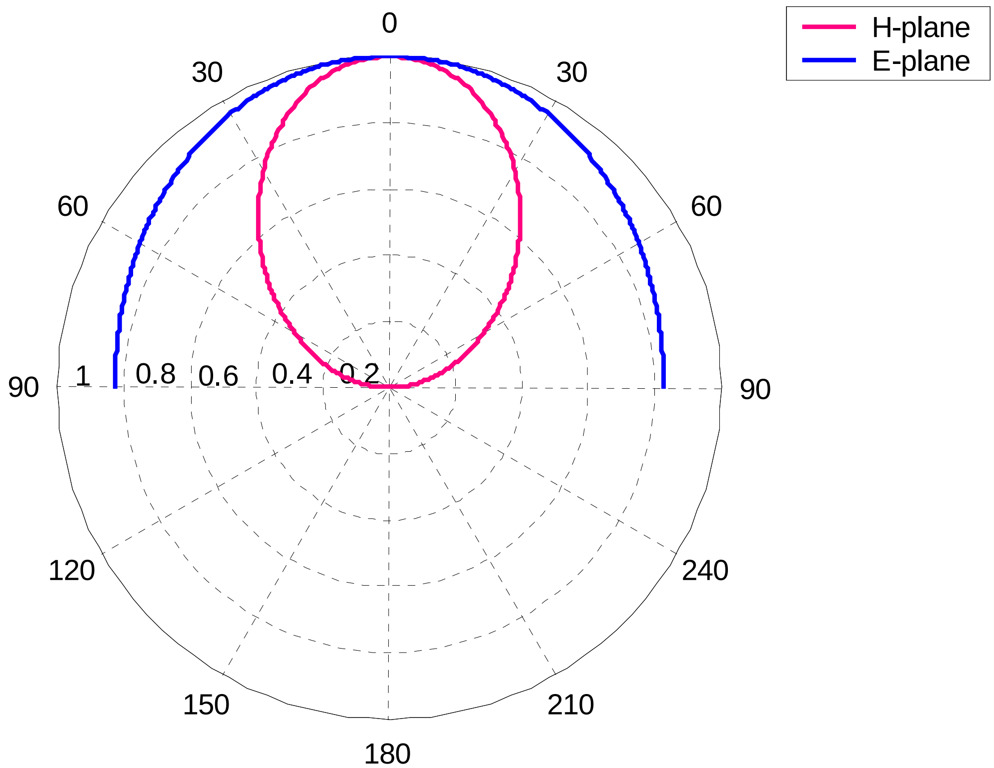
H-plane pattern – uniform vs. tapered illumination ($L_x = 3\lambda$):



The lower SLL of the tapered-source far field is obvious. It is better seen in the rectangular plot given below. The price to pay for the lower SLL is the decrease in directivity (the beamwidth of the major lobe increases).



The above example of $L_x = 3\lambda$ is illustrative on the effect of source distribution on the far-field pattern. However, a more practical example is the rectangular-waveguide open-end aperture, where the waveguide operates in a dominant mode, i.e. $\lambda_0 / 2 < L_x < \lambda_0$. Here, λ_0 is the wavelength in open space ($\lambda_0 = c / f_0$). Consider the case $L_x = 0.75\lambda$. The principal-plane patterns for an aperture on a ground plane look like this:



In the above example, a practical X-band waveguide was considered whose cross-section has the following sizes: $L_x = 2.286$ cm, $L_y = 1.016$ cm. Obviously, $\lambda_0 = 3.048$ cm, and $f_0 = 9.84$ GHz.

The case of a dominant-mode open-end waveguide radiating in free space can be analyzed following the approaches outlined in this Section and in Section 5.

The calculation of beamwidths and directivity is analogous to previous cases. Only the final results will be given here for the case of the *x*-tapered aperture on a ground plane.

Directivity: $$D_0 = \frac{8}{\pi^2}\left(\frac{4\pi}{\lambda^2}L_x L_y\right) \quad (12.81)$$

Effective area: $$A_{eff} = \frac{8}{\pi^2}L_x L_y = 0.81A_p \quad (12.82)$$

Note the decrease in the effective area.

Half-power beamwidths:

$$HPBW_E = \frac{50.6}{L_y / \lambda}, \text{ deg. } (= HPBW_E \text{ of the uniform aperture}) \quad (12.83)$$

$$HPBW_H = \frac{68.8}{L_x / \lambda}, \text{ deg. } (> HPBW_H \text{ of the uniform aperture}) \quad (12.84)$$

The above results are approximate. Better results would be obtained if the following factors were taken into account:

- the phase constant of the waveguide β_g is not equal to the free-space phase constant $\beta_0 = \omega\sqrt{\mu_0\varepsilon_0}$; it is dispersive;
- the abrupt termination at the waveguide open end introduces reflection, which affects the field at the aperture;
- there are strong fringe currents at the waveguide walls, which contribute to the overall radiation.