LECTURE 15: LINEAR ARRAY THEORY - PART I

(*Linear arrays: the two-element array; the N-element array with uniform amplitude and spacing; broad - side array; end-fire array; phased array*)

1. Introduction

Usually the radiation patterns of single-element antennas are relatively wide, i.e. they have relatively low directivity (gain). In long distance communication, antennas with very high directivity are often required. This type of antenna is possible to construct by enlarging the dimensions of the radiating element (maximum size much larger than λ). This approach however may lead to the appearance of multiple side lobes and technologically inconvenient shapes and dimensions. Another way to increase the electrical size of an antenna is to construct it as an assembly of radiating elements in a proper electrical and geometrical configuration – *antenna array*. Usually the array elements are identical. This is not necessary but it is more practical, simple and convenient for design and fabrication. The individual elements may be of any type (wire dipoles or loops, apertures, etc.)

The total field of an array is a vector superposition of the fields radiated by the individual elements. To provide very directive pattern, it is necessary that the partial fields (generated by the individual elements) interfere constructively in the desired direction and interfere destructively in the remaining space.

There are five basic methods to control the overall antenna pattern:
   a) the geometrical configuration of the overall array (linear, circular, spherical, rectangular, etc.)
   b) the relative displacement between elements
   c) the excitation amplitude of individual elements
   d) the excitation phase of each element
   e) the relative pattern of each element

2. Two-element array

Let us represent the electric fields in the far-zone of the array elements in the form:

$$\vec{E}_1 = M_1 E_{n1}(\theta_1, \phi_1) \frac{e^{-j\left(kr_1 - \frac{\beta}{2}\right)}}{r_1} \hat{\rho}_1 \qquad (15.1)$$
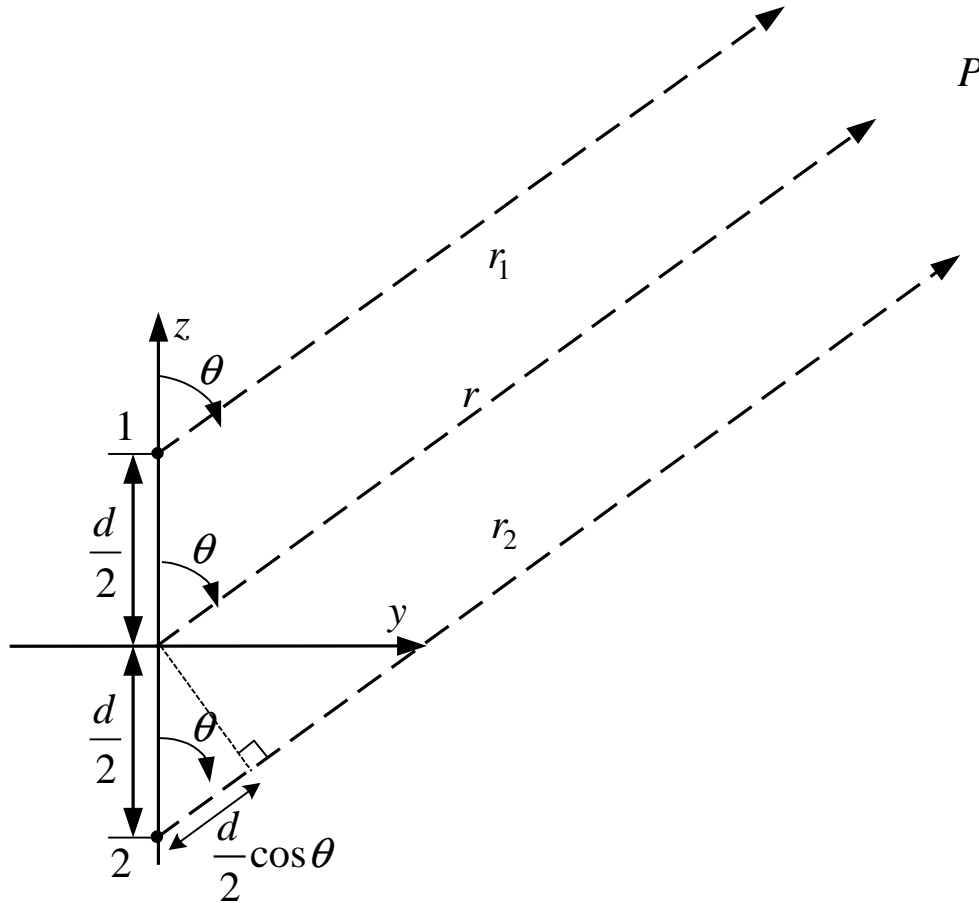
$$\vec{E}_2 = M_2 E_{n2}(\theta_2, \phi_2) \frac{e^{-j\left(kr_2 + \frac{\beta}{2}\right)}}{r_2} \hat{\rho}_2 \qquad (15.2)$$
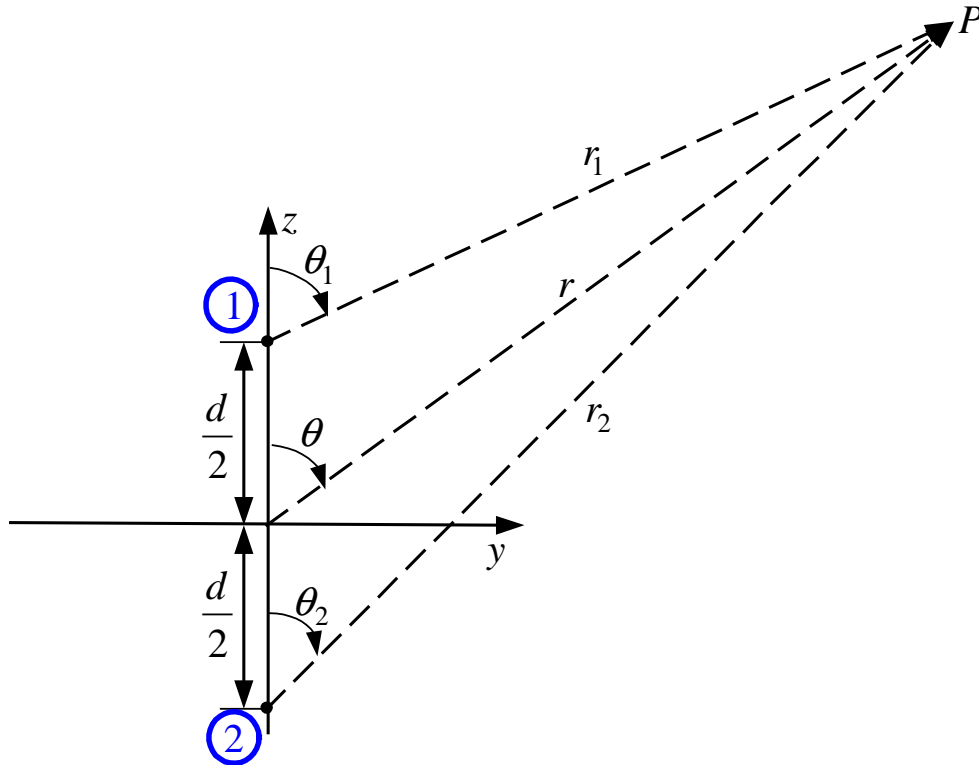
Here:

M_1, M_1	field magnitudes (do not include the $1/r$ factor);
E_{n1}, E_{n2}	normalized field patterns;
r_1, r_2	distances to the observation point P;
β	phase difference between the feed of the two array elements;
$\hat{\rho}_1$, $\hat{\rho}_2$	polarization vectors of the far-zone $\vec{E}$ fields.

The far-field approximation of the two-element array problem:

Let us assume that:
  1) the array elements are identical, i.e.,
$$E_{n1}(\theta,\phi) = E_{n2}(\theta,\phi) = E_n(\theta,\phi) \qquad (15.3)$$
  2) they are oriented in the same way in space (they have identical polarization) i.e.
$$\hat{\rho}_1 = \hat{\rho}_2 = \hat{\rho} \qquad (15.4)$$
  3) their excitation is of the same amplitude, i.e.
$$M_1 = M_2 = M \qquad (15.5)$$

Then, the total field can be derived as:
$$\vec{E} = \vec{E}_1 = \vec{E}_2 \tag{15.6}$$

$$\vec{E} = \hat{\rho} M E_n (\theta, \phi) \frac{1}{r} \left[e^{-jk\left(r - \frac{d}{2}\cos\theta\right) + j\frac{\beta}{2}} + e^{-jk\left(r + \frac{d}{2}\cos\theta\right) - j\frac{\beta}{2}} \right]$$

$$\vec{E} = \hat{\rho} \frac{M}{r} e^{-jkr} E_n (\theta, \phi) \left[e^{j\left(\frac{kd}{2}\cos\theta + \frac{\beta}{2}\right)} + e^{-j\left(\frac{kd}{2}\cos\theta + \frac{\beta}{2}\right)} \right]$$

$$\boxed{\vec{E} = \hat{\rho} M \frac{e^{-jkr}}{r} E_n (\theta, \phi) \times \underbrace{2\cos\left(\frac{kd\cos\theta + \beta}{2} \right)}_{AF}} \tag{15.7}$$

The total field of the array is equal to the product of the field created by a single element located at the origin and a factor called *array factor*, *AF*:
$$AF = 2\cos\left(\frac{kd\cos\theta + \beta}{2} \right) \tag{15.8}$$

Using the normalized field pattern of a single element, $E_n(\theta,\phi)$, and the normalized *AF*,
$$AF_n = \cos\left(\frac{kd\cos\theta + \beta}{2} \right) \tag{15.9}$$

the normalized field pattern of the array can be found as their product:
$$\colorbox{yellow}{$f_n(\theta,\phi) = E_n(\theta,\phi) \times AF_n(\theta,\phi)$} \tag{15.10}$$

The concept illustrated by (15.10) is the so-called *pattern multiplication rule* valid for arrays of identical elements. This rule holds for any array consisting of identical elements, where the excitation magnitudes, the phase shift between elements and displacement between them are not necessarily the same. The total pattern, therefore, can be controlled via the single – element pattern, $E_n(\theta,\phi)$, or via the *AF*.

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The *AF*, in general, depends on:
- number of elements
- geometrical arrangement
- relative excitation magnitudes
- relative phases

Example 1: An array of two horizontal infinitesimal dipoles located at a distance $d = \lambda/4$ from each other. Find the nulls of the total field, if the excitation magnitudes are the same and the phase difference is:

a) $\beta = 0$

b) $\beta = \pi/2$

c) $\beta = -\pi/2$
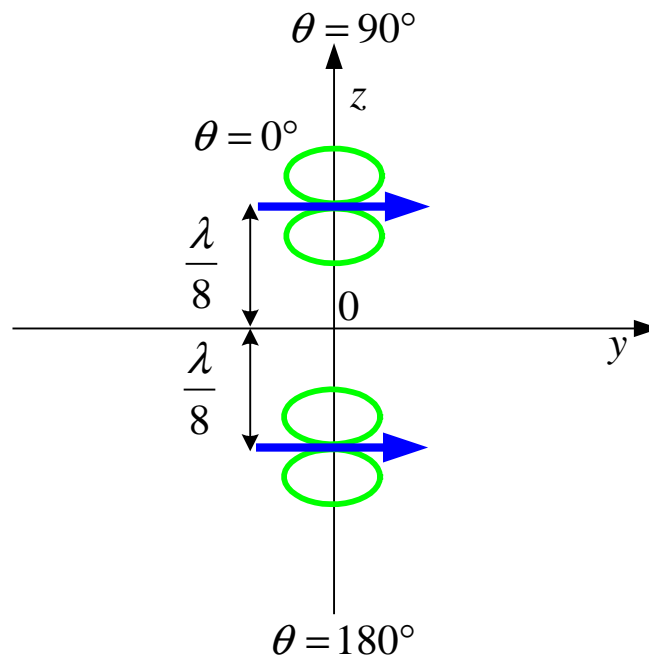
The element factor, $E_n(\theta,\phi)$, does not depend on β and it produces in all three cases one and the same null. Since $E_n(\theta,\phi) = |\cos\theta|$, it is at

$$\theta_1 = \pi/2 \qquad (15.11)$$

The *AF* depends on β and will produce different results in these 3 cases:

a) $\beta = \theta$

$$AF_n = \cos\left(\frac{kd\cos\theta_n}{2}\right) = 0$$

$$\Rightarrow \cos\left(\frac{\pi}{4}\cos\theta_n\right) = 0 \Rightarrow \frac{\pi}{4}\cos\theta_n = \frac{\pi}{2} \Rightarrow \cos\theta_n = 2$$

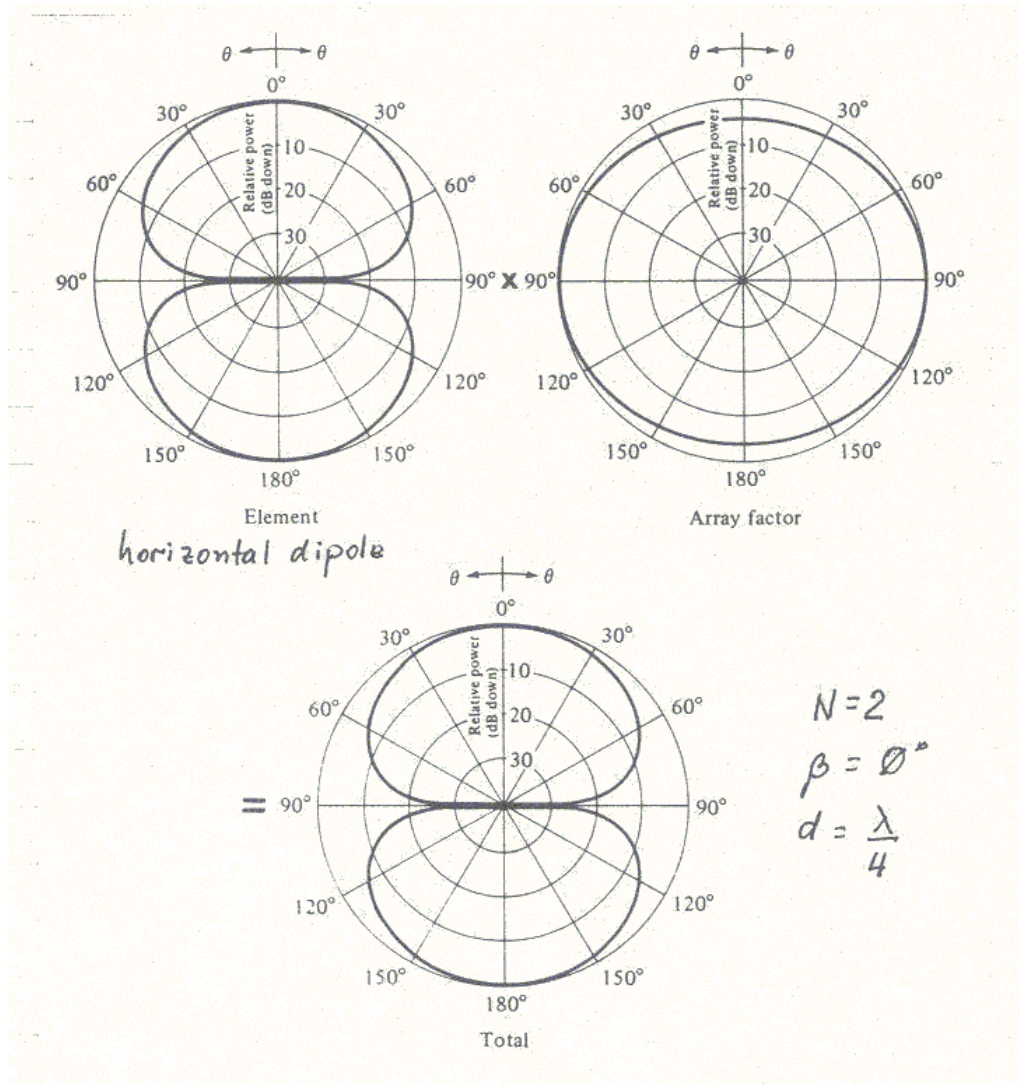The solution does not exist. In this case the total field pattern has only 1 null at $\theta = 90°$.

Element

horizontal dipole

Array factor

Total

$N = 2$

$\beta = \emptyset°$

$d = \dfrac{\lambda}{4}$

Fig 6.3, pp 255, Balanis

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b) $\beta = \dfrac{\pi}{2}$

$$AF_n = \cos\left(\frac{\pi}{4}\cos\theta_n + \frac{\pi}{4}\right) = 0$$

$$\Rightarrow \frac{\pi}{4}\left(\cos\theta_n + 1\right) = \frac{\pi}{2} \;\Rightarrow\; \cos\theta_n + 1 = 2 \;\Rightarrow\; \cos\theta_n = 1 \;\Rightarrow\; \boxed{\theta_2 = 0}$$

The equation

$$\frac{\pi}{4}\left(\cos\theta_n + 1\right) = -\frac{\pi}{2}$$

does not have a solution.

The total field pattern has 2 nulls – at $\theta_1 = 90°$ and at $\theta_2 = 0°$:

Element

horizontal dipole

Array factor

$N=2$

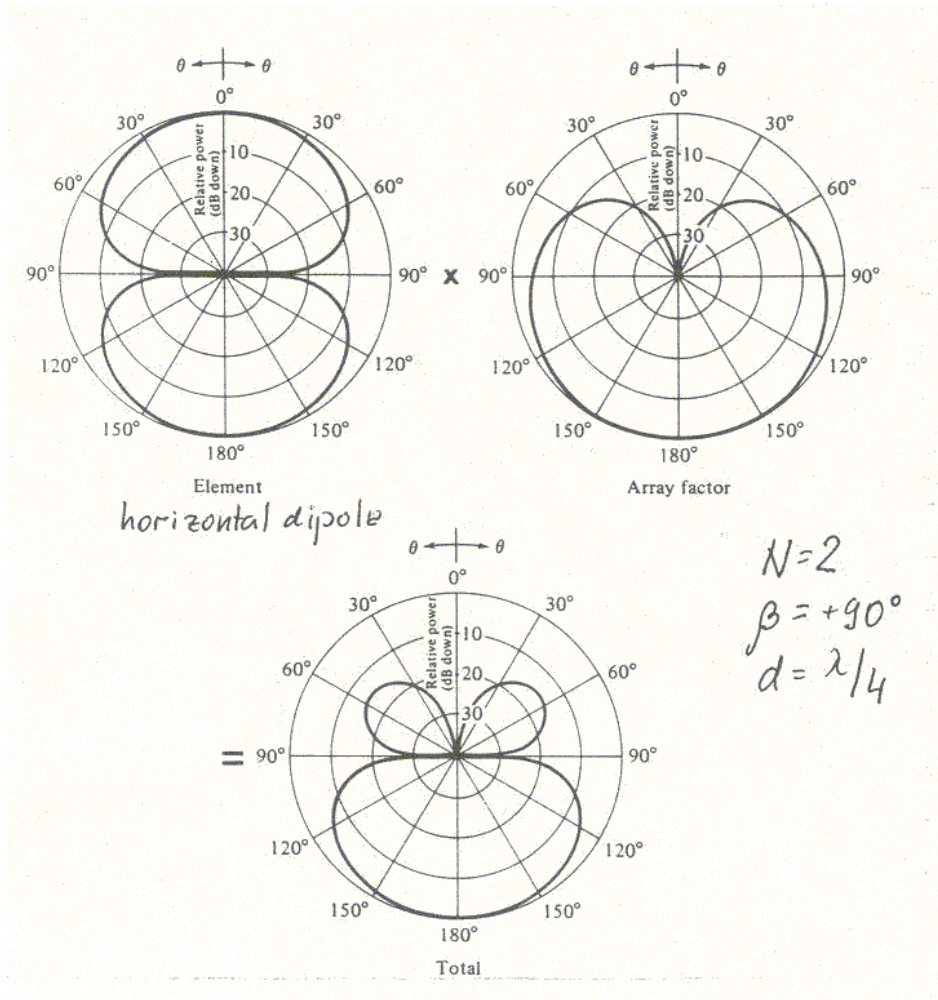$\beta = +90°$

$d = \lambda/4$

Total

Fig 6.4, pp.256, Balanis

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c) $\beta = -\dfrac{\pi}{2}$

$$AF_n = \cos\left(\frac{\pi}{4}\cos\theta_n - \frac{\pi}{4}\right) = 0$$

$$\Rightarrow \frac{\pi}{4}\left(\cos\theta_n - 1\right) = \pm\frac{\pi}{2} \Rightarrow \cos\theta_n - 1 = \pm 2 \Rightarrow \boxed{\theta_2 = \pi}$$

The total field pattern has 2 nulls: at $\theta_1 = 90°$ and at $\theta_2 = 180°$.

Element
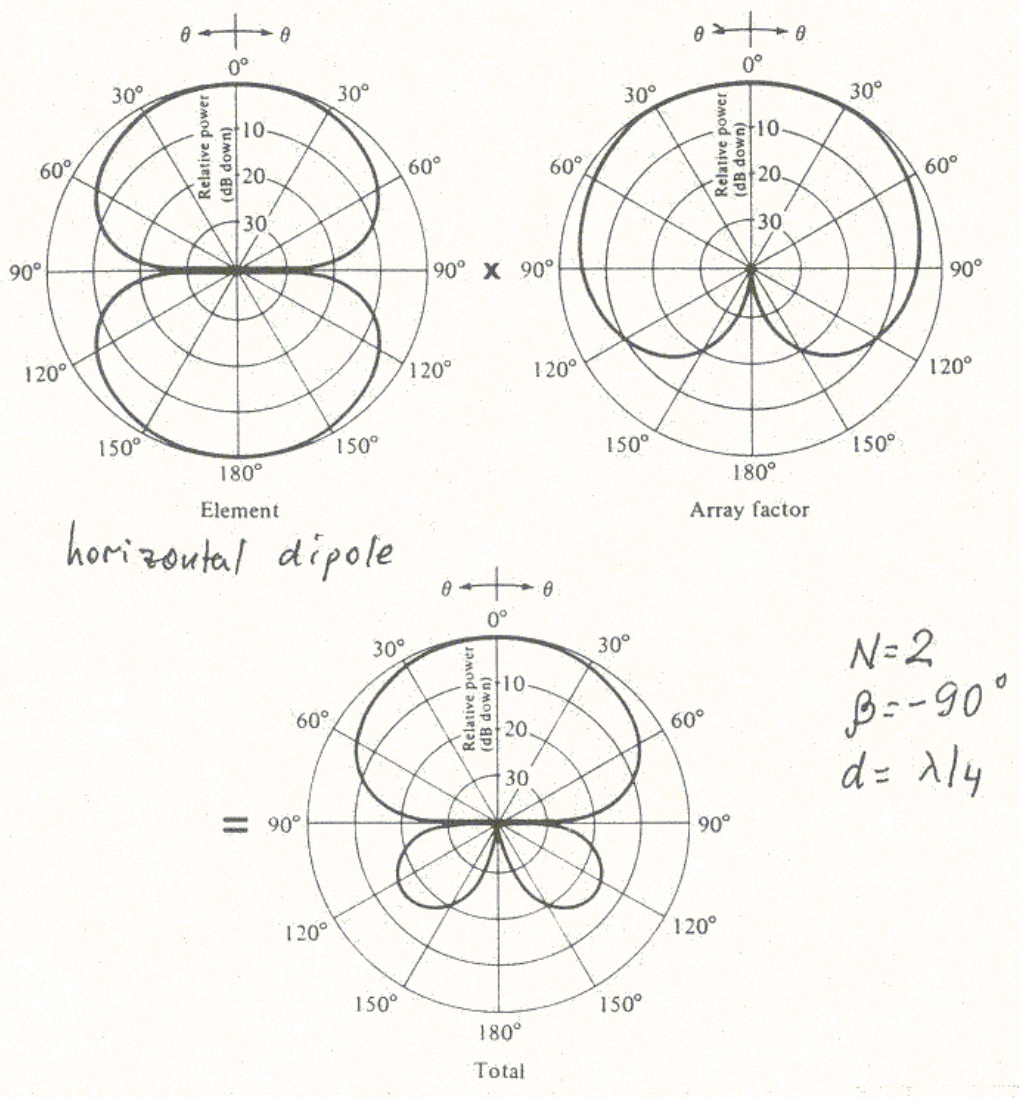
horizontal dipole

Array factor

$N = 2$

$\beta = -90°$

$d = \lambda/4$

Total

<u>Example #2:</u> Consider a 2-element array of two identical dipoles (infinitesimal ones) oriented along the *y*-axis. Find the angles of observation where nulls of the pattern occur as a function of the distance between the dipoles, d, and the phase difference, β.

The normalized total field pattern is:

$$f_n = |\cos\theta| \times \cos\left(\frac{kd\cos\theta + \beta}{2}\right) \qquad (15.12)$$

In order to find the nulls, the equation

$$f_n = |\cos\theta| \cos\left(\frac{kd\cos\theta + \beta}{2}\right) = 0 \qquad (15.13)$$

will be soled.

The element factor, $|\cos\theta|$, produces one null at

$$\theta_1 = \pi/2 \qquad (15.14)$$

The amplitude factor leads to the following solution:

$$\cos\left(\frac{kd\cos\theta + \beta}{2}\right) = 0 \quad \Rightarrow \quad \frac{kd\cos\theta + \beta}{2} = \pm\left(\frac{2n+1}{2}\right)\pi$$

$$\theta_n = \arccos\left[\frac{\lambda}{2\pi d}\left(-\beta \pm (2n+1)\pi\right)\right], n = 0,1,2... \quad (15.15)$$

When there is no phase difference between the two elements $(\beta = 0)$, the separation d must satisfy:

$$d \geq \frac{\lambda}{2}$$

in order at least one null to occur due to (15.15).

3. *N*-element linear array with uniform amplitude and spacing

It is assumed that each succeeding element has a β progressive phase lead current excitation relative to the preceding one. *An array of identical elements with identical magnitudes and with a progressive phase is called a uniform array.*

The *AF* can be obtained by considering the individual elements as point (isotropic) sources. If the elements are of any other pattern, the total field pattern can be obtained by simply multiplying the *AF* by the normalized field pattern of the individual element.

The *AF* of an *N*-element linear array of isotropic sources is:

$$AF = 1 + e^{j(kd\cos\theta+\beta)} + e^{j2(kd\cos\theta+\beta)} + \ldots + e^{j(N-1)(kd\cos\theta+\beta)} \quad (15.16)$$
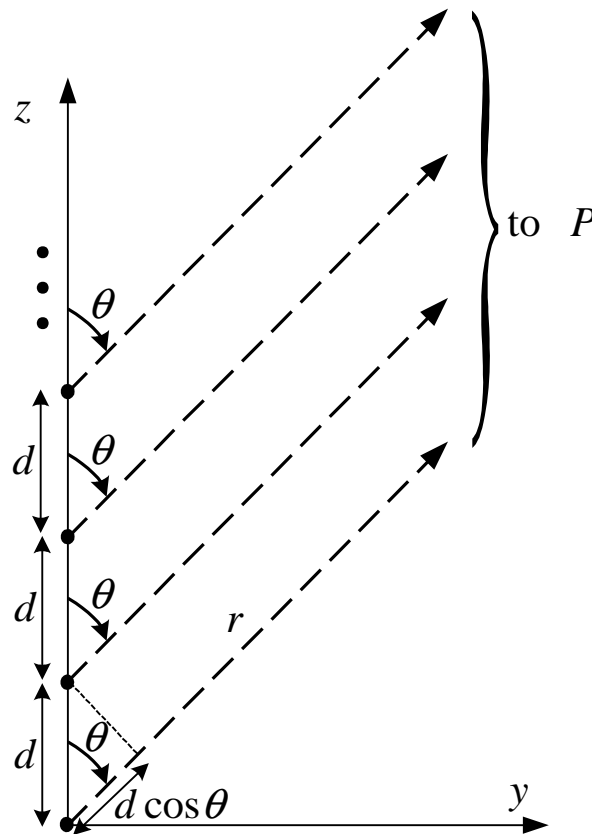
Phase terms of partial fields:

$$1^{st} \rightarrow e^{-jkr}$$

$$2^{nd} \rightarrow e^{-jk(r-d\cos\theta)}$$

$$3^{rd} \rightarrow e^{-jk(r-2d\cos\theta)}$$

$$\ldots$$

$$N^{th} \rightarrow e^{-jk(r-(N-1)d\cos\theta)}$$

Equation (15.16) can be re-written as:

$$AF = \sum_{n=1}^{N} e^{j(n-1)(kd\cos\theta+\beta)} \tag{15.17}$$

$$AF = \sum_{n=1}^{N} e^{j(n-1)\psi} \tag{15.18}$$

where $\psi = kd\cos\theta + \beta$.

From (15.18), it is obvious that the *AF*s of uniform linear arrays can be controlled by the relative phase β between the elements. The *AF* in (15.18) can be expressed in a closed form, which is more convenient for pattern analysis.

$$AF \cdot e^{j\psi} = \sum_{n=1}^{N} e^{jn\psi} \tag{15.19}$$

$$AF \cdot e^{j\psi} - AF = e^{jN\psi} - 1$$

$$AF = \frac{e^{jN\psi} - 1}{e^{j\psi} - 1} = \frac{e^{j\frac{N}{2}\psi}\left(e^{j\frac{N}{2}\psi} - e^{-j\frac{N}{2}\psi}\right)}{e^{j\frac{\psi}{2}}\left(e^{j\frac{\psi}{2}} - e^{-j\frac{\psi}{2}}\right)}$$

$$AF = e^{j\left(\frac{N-1}{2}\right)\psi} \cdot \frac{\sin\left[\dfrac{N}{2}\psi\right]}{\sin\left[\dfrac{\psi}{2}\right]} \tag{15.20}$$

Here, N shows the location of the last element with respect to the reference point in steps with length d. The phase factor $e^{j(N-1)\frac{\psi}{2}}$ is not important unless the array output signal is further combined with the output signal of another antenna. It represents the phase shift of the array's phase centre relative to the origin and it would be identically equal to one if the origin were to coincide with the array's centre. Neglecting the phase factor gives:

$$AF = \frac{\sin\left[\dfrac{N}{2}\psi\right]}{\sin\left[\dfrac{\psi}{2}\right]} \tag{15.21}$$

For small values of ψ, (15.21) can be reduced to:

$$AF = \frac{\sin\left[\dfrac{N}{2}\psi\right]}{\dfrac{\psi}{2}} \tag{15.22}$$
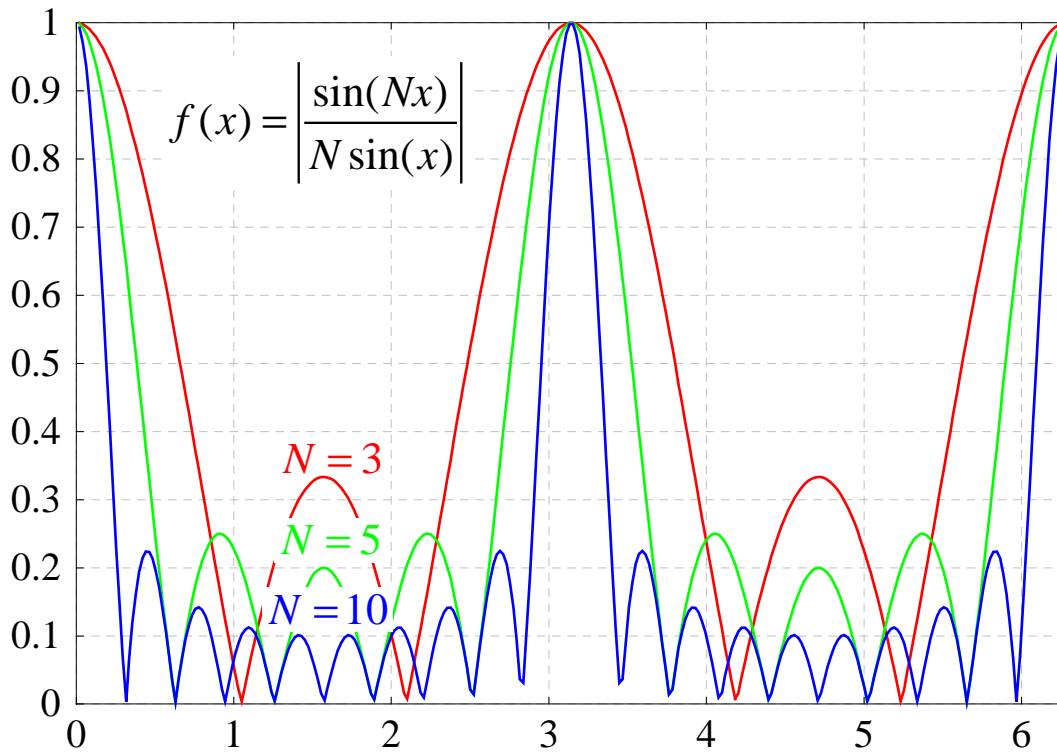
To normalize (15.22) or (15.21) one needs the maximum of the *AF*.

$$AF = N \cdot \frac{\sin\left[\dfrac{N}{2}\psi\right]}{N\sin\left[\dfrac{\psi}{2}\right]} \tag{15.23}$$

The function:
$$f(x) = \frac{\sin(Nx)}{N\sin(x)}$$

has its maximum at $x = 0, \pi, \ldots$, and the value of this maximum is $f_{max} = 1$. Therefore, $AF_{max} = N$.

$$f(x) = \left| \frac{\sin(Nx)}{N\sin(x)} \right|$$

$N = 3$
$N = 5$
$N = 10$

The normalized AF is obtained as:

$$AF_n = \frac{1}{N} \left[\frac{\sin\left[N \frac{\psi}{2} \right]}{N\sin\left[\frac{\psi}{2} \right]} \right] \qquad (15.24)$$

or

$$AF_n = \frac{1}{N} \left[\frac{\sin\left[\frac{N}{2} \psi \right]}{\frac{\psi}{2}} \right], \text{ for small } \psi \qquad (15.25)$$

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Nulls of the *AF*

To find the nulls of the *AF*, equation (15.24) is set equal to zero:

$$\sin\left(\frac{N}{2}\psi\right)=0 \Rightarrow \frac{N}{2}\psi=\pm n\pi \Rightarrow \frac{N}{2}\left(kd\cos\theta_n+\beta\right)=\pm n\pi \quad (15.26)$$

$$\theta_n=\arccos\left[\frac{\lambda}{2\pi d}\left(-\beta\pm\frac{2n}{N}\pi\right)\right], \quad (15.27)$$

$$n=1,2,3\ldots(n\neq 0,N,2N,3N\ldots)$$

When $n=0,N,2N,3N\ldots$, the *AF* attains its maximum values (see the case below). The values of *n* determine the order of the nulls. For a null to exist, *the argument of the arccosine must not exceed unity.*

Maxima of the *AF*

They are studied in order to determine the maximum directivity, the *HPBW's*, the direction of maximum radiation. The maxima of (15.24) occur when (see the plot in page 13):

$$\frac{\psi}{2}=\frac{1}{2}\left(kd\cos\theta_m+\beta\right)=\pm m\pi \quad (15.28)$$

$$\theta_m=\arccos\left[\frac{\lambda}{2\pi d}\left(-\beta\pm 2m\pi\right)\right], \quad m=0,1,2\ldots \quad (15.29)$$

When (15.28) is true, $AF_n=1$, i.e. these are not maxima of minor lobes. The index *m* shows the maximum's order. It is usually desirable to have a single major lobe, i.e. *m=0*. This can be achieved by choosing d/λ sufficiently small. Then, the argument of the arccosine function is (15.29) becomes greater than unity for $m=1,2\ldots$ and equation (15.29) has a single solution:

$$\theta_m=\arccos\left[\frac{-\beta\lambda}{2\pi d}\right] \quad (15.30)$$

The *HPBW* of a major lobe

It can be calculated by setting the value of AF_n equal to $1/\sqrt{2}$. For AF_n in (15.25):
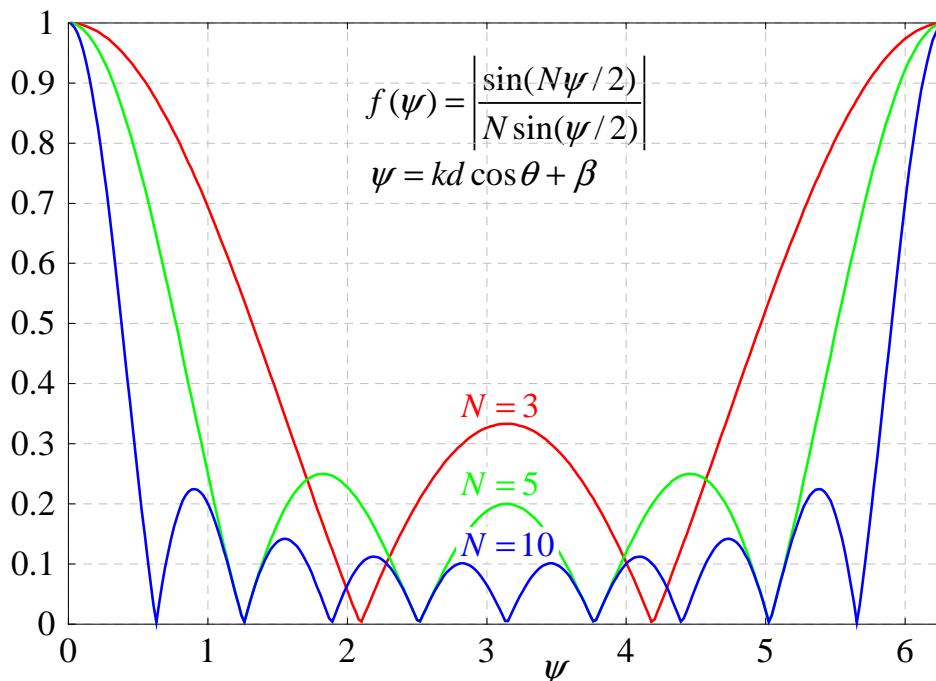
$$\frac{N}{2}\psi = \frac{N}{2}\left(kd\cos\theta_h + \beta\right) = \pm 1.391$$

$$\Rightarrow \theta_h = \arccos\left[\frac{\lambda}{2\pi d}\left(-\beta \pm \frac{2.782}{N}\right)\right] \qquad (15.31)$$

For a symmetrical pattern around θ_m (the angle at which maximum radiation occurs), the *HPBW* ca be calculated as:

$$HPBW = 2\left|\theta_m - \theta_h\right| \qquad (15.32)$$

Maxima of minor lobes (secondary maxima)

They are the maxima of AF_n, where $AF_n \leq 1$. This is clearly seen in the plot of the array factors as a function of $\psi = kd\cos\theta + \beta$ for a uniform equally spaced linear array (*N*=3,5,10).

$$f(\psi) = \left|\frac{\sin(N\psi/2)}{N\sin(\psi/2)}\right|$$

$$\psi = kd\cos\theta + \beta$$

Only approximate solutions will be given here. For the approximated AF_n of equation (15.25), the secondary maxima occur approximately where the numerator attains a maximum:

$$\sin\left(\frac{N}{2}\psi\right) = \pm 1$$

(15.33)

$$\frac{N}{2}(kd\cos\theta + \beta) = \pm\left(\frac{2s+1}{2}\right)\pi$$

$$\theta_s = \arccos\left\{\frac{\lambda}{2\pi d}\left(-\beta \pm \left(\frac{2s+1}{N}\right)\pi\right)\right\} \quad \text{or} \qquad (15.34)$$

$$\theta_s = \frac{\pi}{2} - \arccos\left\{\frac{\lambda}{2\pi d}\left(-\beta \pm \left(\frac{2s+1}{N}\right)\pi\right)\right\} \qquad (15.35)$$

4. Broadside array

A broadside array is an array, which has maximum radiation at $\theta = 90°$ (normal to the axis of the array). For optimal solution both, the element factor and the AF, should have their maxima at $\theta = 90°$.

From (15.28), it follows that the maximum of the AF would occur when

$$\psi = kd\cos\theta + \beta = 0 \qquad (15.36)$$

Equation (15.36) is valid for the zero-th order maximum, $m = 0$. If $\theta_m = \pi/2$, then:

$$\boxed{\psi = \beta = 0} \qquad (15.37)$$

The uniform linear array will have its maximum radiation at $\theta = 90°$, if all array elements have the same phase excitation.

To ensure that there are no maxima in other directions (grating lobes), the separation between the elements should not be equal to multiples of a wavelength:

$$d \neq n\lambda, \quad n = 1,2,3,\ldots \qquad (15.38)$$

If $d \neq n\lambda$, then additional maxima, $AF_n = 1$, appear. Assume that $d = n\lambda$. Then,

$$\psi = kd\cos\theta = \frac{2\pi}{\lambda}n\lambda\cos\theta = 2\pi n\cos\theta \qquad (15.39)$$

If equation (15.39) is true, the maximum condition:

$$\psi_m = 2m\pi, \quad m = 0,\pm 1,\pm 2\ldots \qquad (15.40)$$

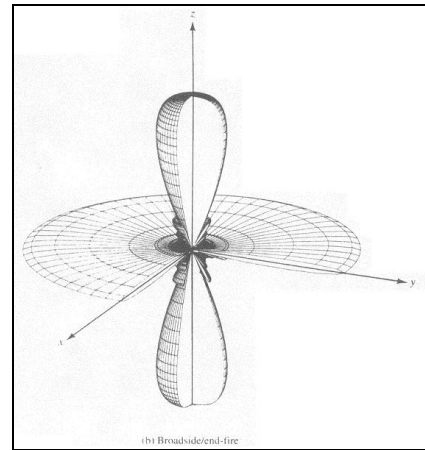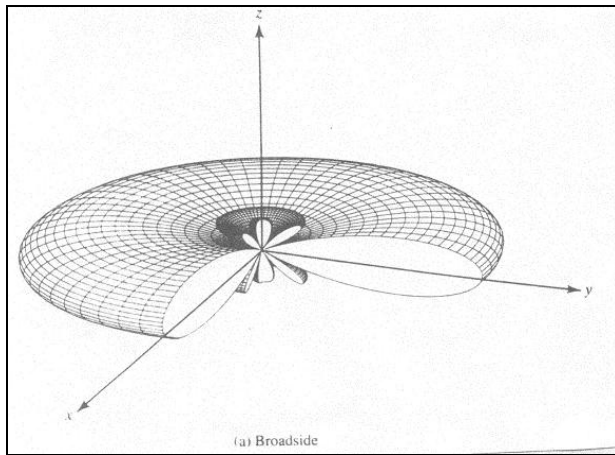will be fulfilled not only for $\theta = \pi/2$ but also for

$$\theta_g = \arccos\left(\frac{m}{n}\right), \quad m = \pm 1,\pm 2\ldots \qquad (15.41)$$

If, for example, $d = \lambda$ $(n = 1)$, equation (15.41) results in two additional major lobes at:

$$\theta_g = \arccos(\pm 1) \Rightarrow \theta_{g_{1,2}} = 0°, 180°$$

If $d = 2\lambda$ $(n = 2)$, equation (15.41) results in four additional major lobes at:

$$\theta_g = \arccos\left(\pm\frac{1}{2}, \pm 1\right) \Rightarrow \theta_{g_{1,2,3,4}} = 0°, 60°, 120°, 180°$$

(a) Broadside   (b) Broadside/end-fire

The best way to ensure the existence of only one maximum is to choose $d_{max} < \lambda$. Then, in the case of the broadside array $(\beta = 0)$, equation (15.29) produces no solution for $m \geq 1$.

5. Ordinary end-fire array

An end-fire array is an array, which has its maximum radiation along the axis of the array ($\theta = 0°, 180°$). It might be required that the array radiates only in one direction – either $\theta = 0°$ or $\theta = 180°$.

$$\psi = kd\cos\theta + \beta\big|_{\theta=0°} = kd + \beta = 0 \ \text{(for max. } AF\text{)}$$

Then:

$$\boxed{\beta = -kd, \ \ \text{for} \ \ \theta_{max} = 0°} \qquad (15.42)$$

$$\psi = kd\cos\theta + \beta\big|_{\theta=180°} = -kd + \beta = 0 \ \text{(for max. } AF\text{)}$$

$$\boxed{\beta = kd, \ \ \text{for} \ \ \theta_{max} = 180°} \qquad (15.43)$$

If the element separation is multiple of a wavelength, $d = n\lambda$, then in addition to the end-fire maxima there also exist maxima in the broadside directions. As with the broadside array, in order to avoid grating lobes, the maximum spacing between the element should be less than λ:

$$d_{max} < \lambda$$

(Show that an end-fire array with $d = \lambda/2$ will have 2 maxima for $\beta = -kd$: at $\theta = 0°$ and at $\theta = 180°$)

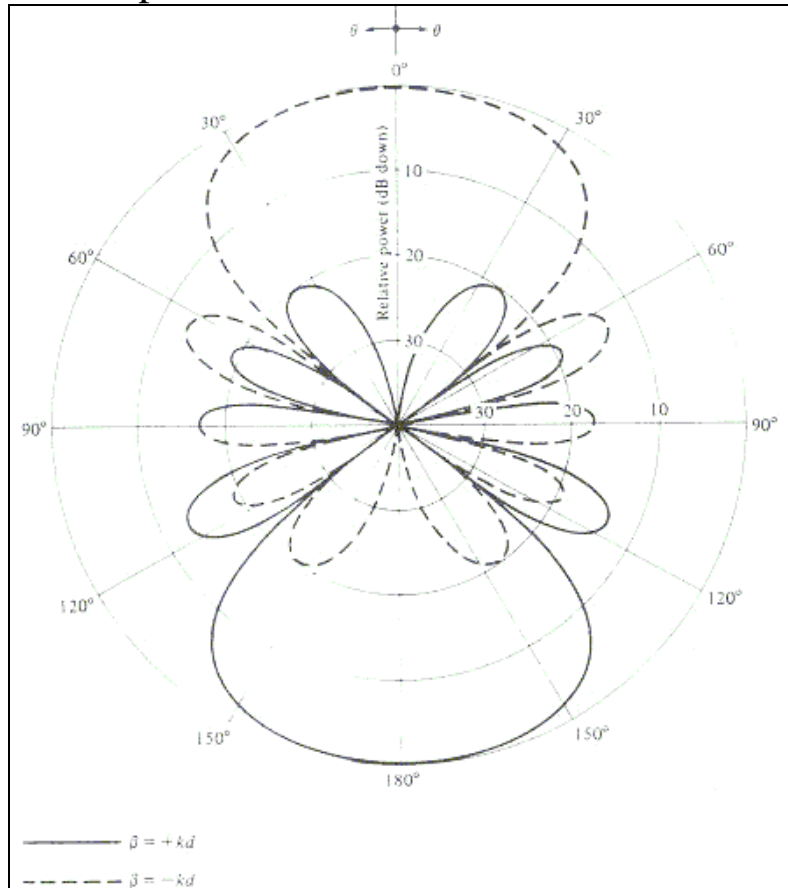*AF* pattern of an *EFA*: N=10, $d = \lambda / 4$

fig 6-11, pp 270, Balanis

6. Phased (scanning) arrays

It was already shown that the zero-th order maximum (*m*=0) of AF_n occurs when

$$\psi = kd \cos\theta_0 + \beta = 0 \qquad (15.44)$$

The relation between the direction of the main beam θ_0 and the phase difference β is obvious. The direction of the main beam can be controlled by the phase shift β. This is the basic principle of *electronic scanning* for phased arrays.

The scanning must be continuous. That is why the feeding system should be capable of continuously varying the progressive phase β between the elements. This is accomplished by ferrite or diode shifters (varactors).

<u>Example</u>: Values of the progressive phase shift β as dependent on the direction of the main beam θ_0 for a uniform linear array with $d = \lambda / 4$.

From equation (15.44):

$$\beta = -kd \cos\theta_0 = -\frac{2\pi}{\lambda}\frac{\lambda}{4}\cos\theta_0 = -\frac{\pi}{2}\cos\theta_0$$

θ_0	β
0°	-90°
60°	-45°
120°	+45°
180°	+90°

The *HPBW* of a scanning array is obtained using eq.(15.31), where $\beta = -kd \cos\theta_0$:

$$\theta_{h_{1,2}} = \arccos\left[\frac{\lambda}{2\pi d}\left(-\beta \pm \frac{2.782}{N}\right)\right] \qquad (15.45)$$

The total beamwidth is:

$$HPBW = \theta_{h1} - \theta_{h2} \qquad (15.46)$$

$$HPBW = \arccos\left[\frac{\lambda}{2\pi d}\left(kd\cos\theta_0 - \frac{2.782}{N}\right)\right] -$$

$$\arccos\left[\frac{\lambda}{2\pi d}\left(kd\cos\theta_0 + \frac{2.782}{N}\right)\right] \qquad (15.47)$$

Since $k = 2\pi / \lambda$:

$$HPBW = \arccos\left[\cos\theta_0 - \frac{2.782}{Nkd}\right]$$
$$- \arccos\left[\cos\theta_0 + \frac{2.782}{Nkd}\right] \qquad (15.48)$$

One can use the substitution $N = (L + d)/d$ to obtain:

$$HPBW = \arccos\left[\cos\theta_0 - 0.443\left(\frac{\lambda}{L+d}\right)\right] - \arccos\left[\cos\theta_0 + 0.443\left(\frac{\lambda}{L+d}\right)\right]$$

(15.49)

Here, L is the length of the array.

Equations (15.48) and (15.49) can be used to calculate the *HPBW* of a broadside array, too ($\theta_0 = 90° = \text{const}$). However, it is not valid *for end-fire arrays*, where

$$HPBW = 2\cos^{-1}\left[1 - \frac{2.782}{Nkd}\right]$$

(15.50)