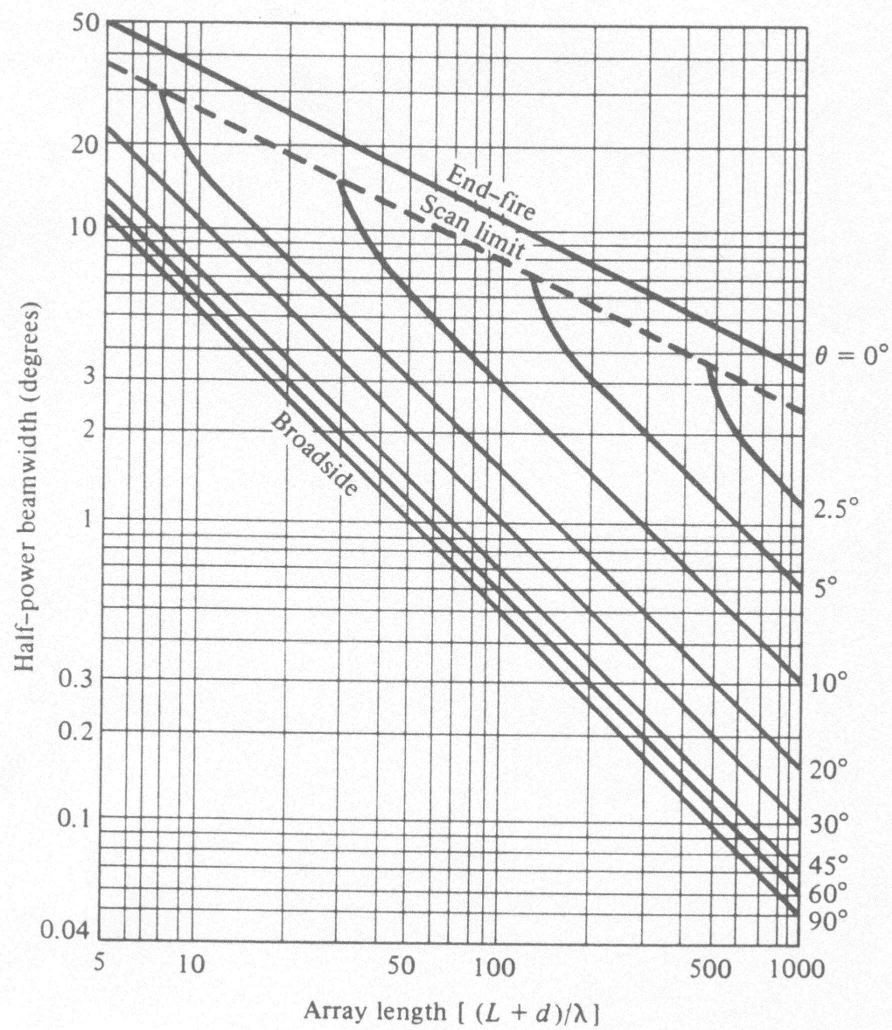
LECTURE 16: LINEAR ARRAY THEORY - PART II

(*Linear arrays: Hansen-Woodyard end-fire array, directivity of a linear array, linear array pattern characteristics – recapitulation; 3-D characteristics of an N-element linear array.*)

1. Hansen-Woodyard end-fire array (HWEFA)

One of the shortcomings of end-fire arrays (EFA) is their relatively broad HPBW as compared to broadside arrays.



Half-power beamwidth for broadside, ordinary end-fire, and scanning uniform linear arrays. (SOURCE: R. S. Elliott, "Beamwidth and Directivity of Large Scanning Arrays," First of Two Parts, *The Microwave Journal*, December 1963)

Fig. 6-11, pp.270, Balanis

To enhance the directivity of an end-fire array, Hansen and Woodyard proposed that the phase shift of an ordinary EFA

$$\beta = \pm kd$$

be increased as:

$$\beta = -\left(kd + \frac{2.94}{N} \right) \text{ for maximum at } \theta = 0^\circ \qquad (16.1)$$

$$\beta = +\left(kd + \frac{2.94}{N} \right) \text{ for maximum at } \theta = 180^\circ \qquad (16.2)$$

Conditions (16.1)–(16.2) are known as the Hansen – Woodyard conditions for end-fire radiation. They follow from a procedure for maximizing the directivity.

The normalized pattern AF_n of a uniform linear array is:

$$AF_n \approx \frac{\sin\left[\frac{N}{2}(kd\cos\theta + \beta) \right]}{\frac{N}{2}(kd\cos\theta + \beta)} \qquad (16.3)$$

if the argument $\psi = \frac{N}{2}(kd\cos\theta + \beta)$ is sufficiently small (see previous lecture). We are looking for optimal β, which would result in maximum directivity. Let:

$$\beta = -pd \qquad (16.4)$$

where d is the array spacing and p is the optimization parameter

$$\Rightarrow AF_n = \frac{\sin\left[\frac{Nd}{2}(k\cos\theta - p) \right]}{\frac{Nd}{2}(k\cos\theta - p)}$$

Assume that $Nd/2 = q$; then

$$\Rightarrow AF_n = \frac{\sin\left[q(kd\cos\theta - p) \right]}{q(kd\cos\theta - p)} \qquad (16.5)$$

or $AF_n = \frac{\sin Z}{Z}$; where $Z = q\left(kd\cos\theta - p\right)$

The radiation intensity:

$$U(\theta) = \left|AF_n\right|^2 = \frac{\sin^2 Z}{Z^2} \qquad (16.6)$$

$$U(\theta = 0) = \left\{\frac{\sin\left[q\left(k - p\right)\right]}{q\left(k - p\right)}\right\}^2 \qquad (16.7)$$

$$U_n(\theta) = \frac{U(\theta)}{U(\theta = 0)} = \left\{\frac{z}{\sin z}\frac{\sin Z}{Z}\right\}^2 \qquad (16.8)$$

where:

$z = q\left(k - p\right)$

$Z = q\left(k\cos\theta - p\right)$, and

$U_n(\theta)$ - normalized power pattern with respect to $\theta = 0°$.

Directivity at $\theta = 0°$:

$$D_0 = \frac{4\pi U(\theta = 0)}{\mathrm{P}_{rad}} \qquad (16.9)$$

where $P_{rad} = \oint\!\!\oint_{\Omega} U_n(\theta) d\Omega$. To maximize the directivity, the quantity $U_0 = P_{rad} / 4\pi$ will be minimized.

$$U_0 = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left(\frac{z}{\sin z}\frac{\sin Z}{Z}\right)^2 \sin\theta d\theta d\theta \qquad (16.10)$$

$$U_0 = \frac{1}{2}\left(\frac{z}{\sin z}\right)^2 \int_0^{\pi}\left\{\frac{\sin\left[q\left(k\cos\theta - p\right)\right]}{q\left(k\cos\theta - p\right)}\right\}^2 \sin\theta d\theta \quad (16.11)$$

$$U_0 = \frac{1}{2kq}\left(\frac{z}{\sin z}\right)^2\left[\frac{\pi}{2} + \frac{\cos 2z - 1}{2z} + \mathrm{Si}(2z)\right] = \frac{1}{2kq} g(z) \quad (16.12)$$

Here, $\mathrm{Si}(z)=\int_0^z \frac{\sin t}{t}dt$. The minimum of $g(z)$ occurs when

$$z=q(k-p)=-1.47 \tag{16.13}$$

$$\Rightarrow \frac{Nd}{2}(k-p)=-1.47$$

$$\Rightarrow \frac{Ndk}{2}-\frac{Ndp}{2}=-1.47, \quad \text{where} \quad dp=-\beta$$

$$\Rightarrow \frac{N}{2}(dk+\beta)=-1.47$$

$$\boxed{\beta=-\frac{2.94}{N}-kd=-\left(\mathrm{kd}+\frac{2.94}{\mathrm{N}}\right)} \tag{16.14}$$

Equation (16.14) gives Hansen-Woodyard condition for improved directivity along $\theta=0°$. Similarly, for $\theta=180°$:

$$\boxed{\beta=+\left(\frac{2.94}{N}+kd\right)} \tag{16.15}$$

Usually, conditions (16.14) and (16.15) are approximated by:

$$\beta=\pm\left(kd+\frac{\pi}{N}\right) \tag{16.16}$$

which is easier to remember and gives almost identical results since the curve $g(z)$ at its minimum is very flat.

Conditions (16.14)-(16.15), or (16.16), ensure minimum beamwidth (maximum directivity) in the desired end-fire direction but there is a trade-off in the side-lobe level, which is higher than that of the ordinary EFA. Besides, conditions (16.14)-(16.15) have to be complemented by additional requirements, which would ensure low level of the radiation in the direction opposite to the main lobe.

a) For a maximum at $\theta = 0°$:

$$\beta = -\left(kd + \frac{2.94}{N}\right)\Big|_{\theta=0°} \Rightarrow \begin{vmatrix} \psi_{\theta=0^\circ} = -\dfrac{2.94}{N} \\ \psi_{\theta=180^\circ} = -2kd - \dfrac{2.94}{N} \end{vmatrix} \tag{16.17}$$

In order to have a minimum of the pattern in the $\theta = 180^\circ$ direction, one must ensure that:

$$|\psi|_{\theta=180°} \simeq \pi, \tag{16.18}$$

It is easier to remember Hansen-Woodyard conditions for maximum directivity in the $\theta = 0°$ direction as:

$$\begin{vmatrix} |\psi|_{\theta=0^\circ} = \dfrac{2.94}{N} \simeq \pi \\ |\psi|_{\theta=180°} \simeq \pi, \end{vmatrix} \tag{16.19}$$

b) For a maximum at $\theta = 180°$:

$$\beta = kd + \frac{2.94}{N}\Big|_{\theta=180°} \Rightarrow \begin{vmatrix} \psi_{\theta=180^\circ} = \dfrac{2.94}{N} \\ \psi_{\theta=0^\circ} = 2kd + \dfrac{2.94}{N} \end{vmatrix} \tag{16.20}$$

In order to have a minimum of the pattern in the $\theta = 0^\circ$ direction, one must ensure that

$$|\psi|_{\theta=0°} \simeq \pi, \tag{16.21}$$

One can now summarize Hansen-Woodyard conditions for maximum directivity in the $\theta = 180°$ direction as:

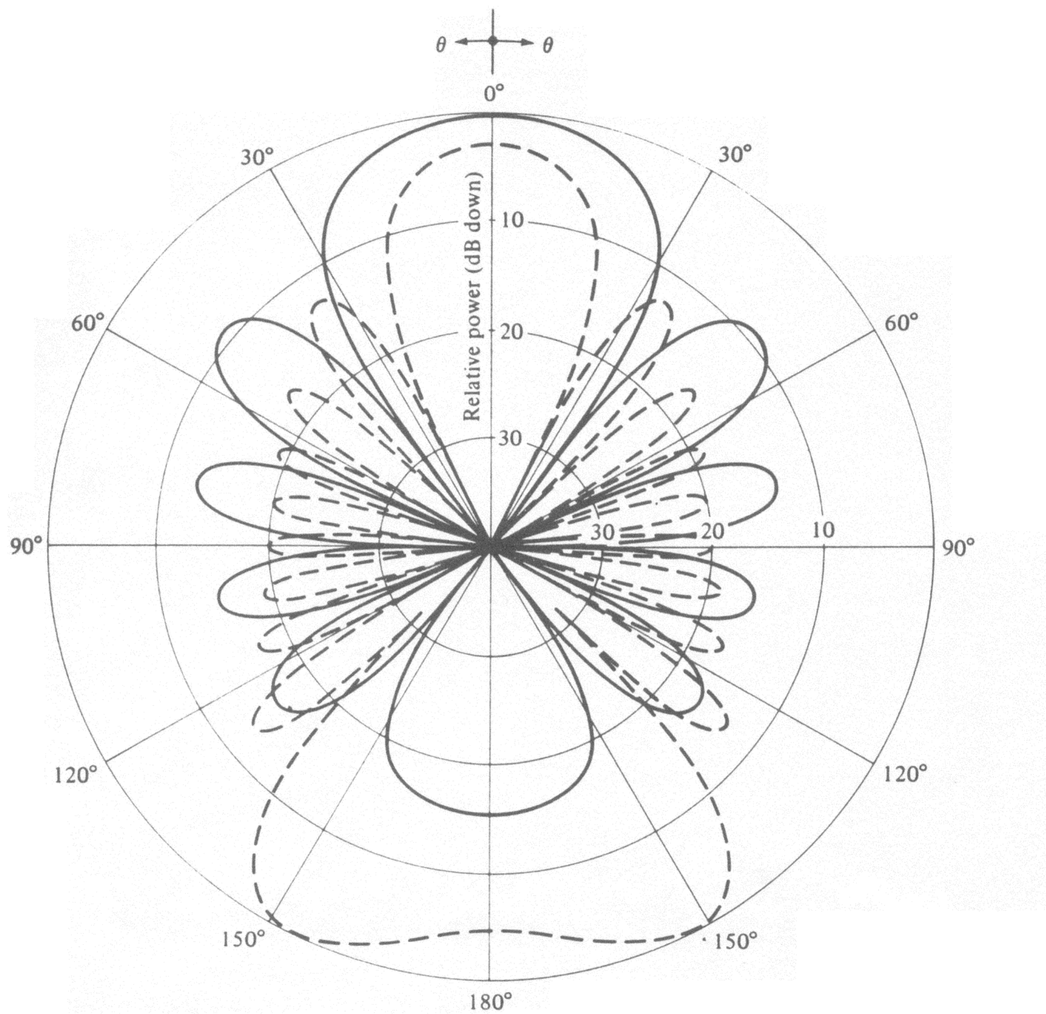
$$\begin{aligned} |\psi|_{\theta=180^\circ} &= \frac{2.94}{N} \simeq \frac{\pi}{N} \\ |\psi|_{\theta=0^\circ} &\simeq \pi \end{aligned} \tag{16.22}$$

If (16.18) and (16.21) are not observed, the radiation in the opposite of the desired direction might even exceed the main beam level. It is easy to show that the complimentary requirement of

$|\psi| = \pi$ at the opposite direction can be met, if the following relation is observed:

$$d = \left(\frac{N-1}{N}\right)\frac{\lambda}{4} \tag{16.23}$$

If *N* is large, $d \simeq \lambda/4$. Thus, for a large uniform array, Hansen-Woodyard condition can yield improved directivity, only if the spacing between the array elements is approximately $\lambda/4$.



Solid line: $d = \lambda/4$
Dotted line: $d = \lambda/2$
N=10
$\beta = -\left(kd + \frac{\pi}{N}\right)$

2. Directivity of a linear array

2.1. Directivity of a BSA

$$U(\theta) = \left|AF_n\right|^2 = \left[\frac{\sin\left(\frac{N}{2}kd\cos\theta\right)}{\frac{N}{2}kd\cos\theta}\right]^2 = \left[\frac{\sin Z}{Z}\right]^2 \quad (16.24)$$

$$D_0 = 4\pi\frac{U_0}{P_{rad}} = \frac{U_0}{U_{av}} \quad (16.25)$$

where:

$$U_{av} = \frac{P_{rad}}{4\pi}$$

The radiation intensity in the direction of maximum radiation $\theta = \pi/2$ in terms of AF_n is unity:

$$U_0 = U_{\max} = U\left(\theta = \pi/2\right) = 1$$

$$\Rightarrow D_0 = \frac{1}{U_{av}} \quad (16.26)$$

The radiation intensity averaged over all directions is calculated as:

$$U_{av} = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{\sin^2 Z}{Z^2}\sin\theta d\theta d\phi = \frac{1}{2}\int_0^{\pi}\left|\frac{\sin\left(\frac{N}{2}kd\cos\theta\right)}{\frac{N}{2}kd\cos\theta}\right|^2 \sin\theta d\theta$$

Change variable:

$$Z = \frac{N}{2}kd\cos\theta \Rightarrow dZ = -\frac{N}{2}kd\cos\theta d\theta \quad (16.27)$$

$$U_{av} = -\frac{1}{2}\int_0^{\pi}\left[\frac{\sin Z}{Z}\right]^2 d\cos\theta d\theta = -\frac{1}{2N}\frac{2}{kd}\int_{\frac{Nkd}{2}}^{-\frac{Nkd}{2}}\left(\frac{\sin Z}{Z}\right)^2 dZ \quad (16.28)$$

$$U_{av} = \frac{1}{Nkd} \int_{-\frac{Nkd}{2}}^{\frac{Nkd}{2}} \left(\frac{\sin Z}{Z} \right)^2 dZ \tag{16.29}$$

The function, $\left(\frac{\sin Z}{Z} \right)^2$, is a relatively fast decaying function as Z increases. That is why, for large arrays, where $Nkd/2$ is big enough (≥ 20), the integral (16.29) can be approximated by:

$$U_{av} \simeq \frac{1}{Nkd} \int_{-\infty}^{\infty} \left(\frac{\sin Z}{Z} \right)^2 dZ = \frac{\pi}{Nkd} \tag{16.30}$$

$$D_0 = \frac{1}{U_{av}} \simeq \frac{Nkd}{\pi} \simeq 2N \left(\frac{d}{\lambda} \right) \tag{16.31}$$

Substituting the length of the array $L = (N-1)d$ in (16.31) yields:

$$D_0 = 2 \underbrace{\left(1 + \frac{L}{d} \right)}_{N} \left(\frac{d}{\lambda} \right) \tag{16.32}$$

For a large array $(L \gg d)$:

$$D_0 \simeq 2L/\lambda \tag{16.33}$$

2.1 Directivity of ordinary EFA

Consider an EFA with maximum radiation at $\theta = 0^\circ$, i.e. $\beta = -kd$.

$$U(\theta) = |AF_n|^2 = \left\{ \frac{\sin\left[\frac{N}{2} kd(\cos\theta - 1) \right]}{\left[\frac{N}{2} kd(\cos\theta - 1) \right]} \right\}^2 = \left(\frac{\sin Z}{Z} \right)^2 \tag{16.34}$$

where: $Z = \frac{N}{2} kd(\cos\theta - 1)$.

$$U_{av} = \frac{P_{rad}}{4\pi} = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left(\frac{\sin Z}{Z}\right)^2 \sin\theta d\theta d\phi = \frac{1}{2}\int_0^{\pi}\left(\frac{\sin Z}{Z}\right)^2 \sin\theta d\theta$$

Again, change of variables is used:

$$Z = \frac{N}{2}kd\cos\theta \Rightarrow dZ = -\frac{N}{2}kd\cos\theta d\theta \qquad (16.35)$$

$$U_{av} = -\frac{1}{2}\int_0^{\pi}\left(\frac{\sin Z}{Z}\right)^2 d\cos\theta = -\frac{1}{2}\frac{2}{Nkd}\int_0^{-Nkd}\left(\frac{\sin Z}{Z}\right)^2 dZ$$

$$U_{av} = \frac{1}{Nkd}\int_0^{Nkd}\left(\frac{\sin Z}{Z}\right)^2 dZ \qquad (16.36)$$

If (Nkd) is sufficiently large, the above integral can be approximated as:

$$U_{av} = \frac{1}{Nkd}\int_0^{\infty}\left(\frac{\sin Z}{Z}\right)^2 dZ = \frac{1}{Nkd}\frac{\pi}{2} \qquad (16.37)$$

$$\Rightarrow D_0 \simeq \frac{1}{U_{av}} = \frac{2Nkd}{\pi} = 4N\left(\frac{d}{\lambda}\right) \qquad (16.38)$$

It is seen that the directivity of an EFA is approximately twice as large as the directivity of the BSA (compare (16.38) and (16.31)). Another (equivalent) expression can be derived for D_0 in terms of the array length $L = (N-1)d$:

$$D_0 = 4\left(1+\frac{L}{d}\right)\left(\frac{d}{\lambda}\right) \qquad (16.39)$$

For large arrays, the following approximation holds:

$$D_0 = 4L/\lambda \quad \text{if} \quad L \gg d \qquad (16.40)$$

2.2 Directivity of HW EFA

If the radiation has its maximum at $\theta = 0°$, then the minimum of U_{av} was obtained as (16.12):

$$U_{av}^{\min} = \frac{1}{2k}\frac{2}{Nd}\left[\frac{Z_{\min}}{\sin Z_{\min}}\right]^2\left[\frac{\pi}{2} + \frac{\cos(2Z_{\min})-1}{2Z_{\min}} + \mathrm{Si}(2Z_{\min})\right] \quad (16.41)$$

where:

$$Z_{\min} = -1.47 \simeq -\frac{\pi}{2}$$

$$\Rightarrow U_{av}^{\min} = \frac{1}{Nkd}\left(\frac{\pi}{2}\right)^2\left[\frac{\pi}{2} + \frac{2}{\pi} - 1.8515\right] = \frac{0.878}{Nkd} \quad (16.42)$$

$$D_0 = \frac{1}{U_{av}^{\min}} = \frac{Nkd}{0.878} = 1.789\left[4N\left(\frac{d}{\lambda}\right)\right] \quad (16.43)$$

From (16.43), one can see that using HW conditions leads to improvement of the directivity of the EFA with a factor of 1.789. Equation (16.43) can be expressed via the length L of the array as:

$$D_0 = 1.789\left[4\left(1+\frac{L}{d}\right)\left(\frac{d}{\lambda}\right)\right] = 1.789\left[4\left(\frac{L}{\lambda}\right)\right] \quad (16.44)$$

<u>Example</u>: Given a linear uniform array of N isotropic elements (N=10), find the directivity D_0 if:

a) $\beta = 0$ (BSA)

b) $\beta = -kd$ (Ordinary EFA)

c) $\beta = -kd - \dfrac{\pi}{N}$ (Hansen-Woodyard EFA)

In all cases, $d = \lambda/4$.

---

a) BSA

$$D_0 \simeq 2N\left(\frac{d}{\lambda}\right) = 5 \ \ (6.999 \ \text{dB})$$

b) Ordinary EFA

$$D_0 \simeq 4N\left(\frac{d}{\lambda}\right) = 10 \quad (10 \text{ dB})$$

c) HW EFA

$$D_0 \simeq 1.789\left[4N\left(\frac{d}{\lambda}\right)\right] = 17.89 \quad (12.53 \text{ dB})$$

---

3. Pattern characteristics of linear uniform arrays - recapitulation
   A. <u>Broad-side array</u>

NULLS ($AF_n = 0$):

$$\theta_n = \arccos\left(\pm\frac{n}{N}\frac{\lambda}{d}\right), \text{ where } \ n = 1,2,3,4,\ldots \text{ and } n \neq N, 2N, 3N, \ldots$$

MAXIMA ($AF_n = 1$):

$$\theta_n = \arccos\left(\pm\frac{m\lambda}{d}\right), \text{ where } \ m = 0,1,2,3,\ldots$$

HALF-POWER POINTS:

$$\theta_h \simeq \arccos\left(\pm\frac{1.391\lambda}{\pi N d}\right), \text{ where } \frac{\pi \mathrm{d}}{\lambda} \ll 1$$

HALF-POWER BEAMWIDTH:

$$\triangle\theta_h = 2\left[\frac{\pi}{2} - \arccos\left(\frac{1.391\lambda}{\pi N d}\right)\right], \quad \frac{\pi d}{\lambda} \ll 1$$

MINOR LOBE MAXIMA:

$$\theta_s \simeq \arccos\left[\pm\frac{\lambda}{2d}\left(\frac{2s+1}{N}\right)\right], \text{ where } s = 1,2,3,\ldots \text{ and } \frac{\pi \mathrm{d}}{\lambda} \ll 1$$

FIRST-NULL BEAMWIDTH (FNBW):

$$\triangle\theta_n = 2\left[\frac{\pi}{2} - \arccos\left(\frac{\lambda}{Nd}\right)\right]$$

FIRST SIDE LOBE BEAMWIDH (FSLBW):

$$\triangle\theta_s = 2\left[\frac{\pi}{2} - \arccos\left(\frac{3\lambda}{2Nd}\right)\right], \quad \frac{\pi d}{\lambda} \ll 1$$

B. <u>Ordinary end-fire array</u>

NULLS ($AF_n = 0$):

$$\theta_n = \arccos\left(1 - \frac{n}{N}\frac{\lambda}{d}\right), \text{ where } n = 1,2,3,\ldots \text{ and } n \neq N, 2N, 3N, \ldots$$

MAXIMA ($AF_n = 1$):

$$\theta_n = \arccos\left(1 - \frac{m\lambda}{d}\right), \text{ where } m = 0,1,2,3,\ldots$$

HALF-POWER POINTS:

$$\theta_h = \arccos\left(1 - \frac{1.391\lambda}{\pi Nd}\right), \text{ where } \frac{\pi \mathrm{d}}{\lambda} \ll 1$$

HALF-POWER BEAMWIDTH:

$$\triangle\theta_h = 2\arccos\left(1 - \frac{1.391\lambda}{\pi Nd}\right), \quad \frac{\pi d}{\lambda} \ll 1$$

MINOR LOBE MAXIMA:

$$\theta_s = \arccos\left[1 - \frac{(2s+1)\lambda}{2Nd}\right], \text{ where } s = 1,2,3,\ldots \text{ and } \frac{\pi \mathrm{d}}{\lambda} \ll 1$$

FIRST-NULL BEAMWIDTH:

$$\triangle\theta_n = 2\arccos\left(1-\frac{\lambda}{Nd}\right)$$

FIRST SIDE LOBE BEAMWIDH:

$$\triangle\theta_s = 2\arccos\left(1-\frac{3\lambda}{2Nd}\right), \quad \frac{\pi d}{\lambda} \ll 1$$

C. <u>Hansen-Woodyard end-fire array</u>

NULLS:

$$\theta_n = \arccos\left[1+(1-2n)\frac{\lambda}{2Nd}\right], \text{ where } n = 1, 2, \dots \text{ and } n \neq N, 2N, \dots$$

MINOR LOBE MAXIMA:

$$\theta_s = \arccos\left(1-\frac{s\lambda}{Nd}\right), \text{ where } s = 1, 2, 3, \dots \text{ and } \frac{\pi \mathrm{d}}{\lambda} \ll 1$$

SECONDARY MAXIMA:

$$\theta_m = \arccos\left\{1+[1-(2m+1)]\frac{\lambda}{2Nd}\right\}, \text{ where } m = 1, 2, \dots \text{ and } \frac{\pi d}{\lambda} \ll 1$$

HALF-POWER POINTS:

$$\theta_h = \arccos\left(1-0.1398\frac{\lambda}{Nd}\right), \text{ where } \frac{\pi \mathrm{d}}{\lambda} \ll 1, \ N\text{-large}$$

HALF-POWER BEAMWIDTH:

$$\triangle\theta_h = 2\arccos\left(1-0.1398\frac{\lambda}{Nd}\right), \text{ where } \frac{\pi d}{\lambda} \ll 1, \ N\text{-Large}$$

FIRST-NULL BEAMWIDTH:

$$\triangle\theta_n = 2\arccos\left(1-\frac{\lambda}{2Nd}\right)$$

4. 3-D characteristics of a linear array

In the previous considerations, it was always assumed that the linear-array elements are located along the *z*-axis, thus, creating a problem, symmetrical around the *z*-axis. If the array axis has an arbitrary orientation, the array factor can be expressed as:

$$AF = \sum_{n=1}^{N} a_n e^{j(n-1)(kd\cos\gamma+\beta)} = \sum_{n=1}^{N} a_n e^{j(n-1)\psi}, \qquad (16.45)$$

where a_n is the excitation amplitude and $\psi = kd\cos\gamma + \beta$.

The angle γ is subtended between the array axis and the radius-vector to the observation point. Thus, if the array axis is along the unit vector $\hat{a}$:

$$\hat{a} = \sin\theta_a\cos\phi_a\hat{x} + \sin\theta_a\sin\phi_a\hat{y} + \cos\theta_a\hat{z} \qquad (16.46)$$

and the radius – vector to the observation point is:

$$\hat{r} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z} \qquad (16.47)$$

the angle γ can be found from:

$$\cos\gamma = \hat{a}\cdot\hat{r}$$

$$= \sin\theta\cos\phi\sin\theta_a\cos\phi_a\hat{x} + \sin\theta\sin\phi\sin\theta_a\sin\phi_a\hat{y} + \cos\theta\cos\theta_a\hat{z}$$

$$\Rightarrow \cos\gamma = \sin\theta\sin\theta_a\cos(\phi-\phi_a) + \cos\theta\cos\theta_a \qquad (16.48)$$

If $\hat{a} = \hat{z}$ $(\theta_a = 0°)$, then $\cos\gamma = \cos\theta$, $\gamma = \theta$.