1. Planar arrays

Planar arrays are more versatile; they provide more symmetrical patterns with lower side lobes, much higher directivity (narrow main beam). They can be used to scan the main beam toward any point in space.

Applications – tracking radars, remote sensing, communications, etc.

1.1 The array factor of a rectangular planar array

Fig. 6.23(b), pp.310, Balanis
The AF of a linear array of $M$ elements along the $x$-axis is:

$$AF_{x1} = \sum_{m=1}^{M} I_{m1} e^{j(m-1)(kd \sin \theta \cos \phi + \beta_x)} \quad (18.1)$$

where $\sin \theta \cos \phi = \cos \gamma_x$ is the directional cosine with respect to the $x$-axis. It is assumed that all elements are equispaced with an interval of $d_x$ and a progressive shift $\beta_x$. $I_{m1}$ denotes the excitation amplitude of the element at the point with coordinates: $x = (m-1)d_x$, $y = 0$. In the figure above, this is the element of the $m$-th row and the 1st column of the array matrix.

If $N$ such arrays are placed next to each other in the $y$ direction, a rectangular array will be formed. We shall assume again that they are equispaced at a distance of $d_y$ and there is a progressive phase shift along each row of $\beta_y$. It will be also assumed that the normalized current distribution along each of the $x$-directed array is the same but the absolute values correspond to a factor of $I_{1n}$ ($n = 1, \ldots, N$). Then, the AF of the entire array will be:

$$AF = \sum_{n=1}^{N} I_{1n} \left[ \sum_{m=1}^{M} I_{m1} e^{j(m-1)(kd \sin \theta \cos \phi + \beta_x)} \right] e^{j(n-1)(kd \sin \theta \cos \phi + \beta_y)} \quad (18.2)$$

or

$$AF = S_{xM} \cdot S_{yN}, \quad (18.3)$$

where:

$$S_{xM} = AF_{x1} = \sum_{m=1}^{M} I_{m1} e^{j(m-1)(kd \sin \theta \cos \phi + \beta_x)}, \quad \text{and}$$

$$S_{yN} = AF_{1y} = \sum_{n=1}^{N} I_{1n} e^{j(n-1)(kd \sin \theta \sin \phi + \beta_y)}$$

In the array factors above:

$$\sin \theta \cos \phi = \hat{x} \cdot \hat{r} = \cos \gamma_x$$

$$\sin \theta \sin \phi = \hat{y} \cdot \hat{r} = \cos \gamma_y \quad (18.4)$$

The pattern of a rectangular array is the product of the array factors of the linear arrays in the $x$ and $y$ directions.
For a uniform planar (rectangular) array $I_{m1} = I_{1n} = I_0$, for all $m$ and $n$, i.e., all elements have the same excitation amplitudes.

$$AF = I_0 \sum_{n=1}^{N} e^{j(m-1)(kd_x \sin \theta \cos \phi + \beta_x)} \sum_{m=1}^{N} e^{j(n-1)(kd_y \sin \theta \sin \phi + \beta_y)}$$  \hspace{1cm} (18.5)

The normalized array factor can be obtained as:

$$AF_n(\theta, \phi) = \left\{ \frac{1}{M} \sin \left( \frac{M \psi_x}{2} \right) \right\} \left\{ \frac{1}{N} \sin \left( \frac{N \psi_y}{2} \right) \right\}$$  \hspace{1cm} (18.6)

where:

$$\psi_x = kd_x \sin \theta \cos \phi + \beta_x$$

$$\psi_y = kd_y \sin \theta \sin \phi + \beta_y$$

The major lobe (principal maximum) and grating lobes of the terms:

$$S_{xM} = \frac{1}{M} \frac{\sin \left( M \frac{\psi_x}{2} \right)}{\sin \left( \frac{\psi_x}{2} \right)}$$  \hspace{1cm} (18.7)

$$S_{yN} = \frac{1}{N} \frac{\sin \left( N \frac{\psi_y}{2} \right)}{\sin \left( \frac{\psi_y}{2} \right)}$$  \hspace{1cm} (18.8)

are located at angles such that:

$$kd_x \sin \theta_m \cos \phi_m + \beta_x = \pm 2m\pi, \hspace{0.5cm} m = 0,1,\ldots$$  \hspace{1cm} (18.9)

$$kd_y \sin \theta_n \sin \phi_n + \beta_y = \pm 2n\pi, \hspace{0.5cm} n = 0,1,\ldots$$  \hspace{1cm} (18.10)

The principal maxima correspond to $m = 0, n = 0$. 
In general, $\beta_x$ and $\beta_y$ are independent from each other. But, if it is required that the main beams of $S_{x,m}$ and $S_{y,n}$ intersect (which is usually the case), then the common main beam is in the direction:

$$\theta = \theta_0 \text{ and } \phi = \phi_0, \ m = n = 0$$

(18.11)

If the principal maximum is specified by $(\theta_0, \phi_0)$, then the progressive phases $\beta_x$ and $\beta_y$ must satisfy:

$$\beta_x = -kd_x \sin \theta_0 \cos \phi_0 \quad (18.12)$$

$$\beta_y = -kd_y \sin \theta_0 \sin \phi_0 \quad (18.13)$$

When $\beta_x$ and $\beta_y$ are specified, the direction of the main beam can be found by simultaneously solving (18.12) and (18.13):

$$\tan \phi_0 = \frac{\beta_y d_x}{\beta_x d_y}$$

(18.14)

$$\sin \theta_0 = \pm \sqrt{\left(\frac{\beta_x}{kd_x}\right)^2 + \left(\frac{\beta_y}{kd_y}\right)^2}$$

(18.15)

The grating lobes can be located by substituting (18.12) and (18.13) in (18.9) and (18.10):

$$\tan \phi_{mn} = \frac{\sin \theta_0 \sin \phi_0 \pm \frac{n\lambda}{d_y}}{\sin \theta_0 \cos \phi_0 \pm \frac{m\lambda}{d_x}}$$

(18.16)

$$\sin \theta_{mn} = \frac{\sin \theta_0 \cos \phi_0 \pm \frac{m\lambda}{d_x}}{\cos \phi_{mn}} = \frac{\sin \theta_0 \sin \phi_0 \pm \frac{n\lambda}{d_y}}{\sin \phi_{mn}}$$

(18.17)
To avoid grating lobes, the spacing between the elements must be less than $\lambda$ ($d_y < \lambda$ and $d_y < \lambda$). In order a true grating lobe to occur, both equations (18.16) and (18.17) must have a real solution $(\theta_{mn}, \phi_{mn})$.

3-D pattern of a 5-element square planar uniform array without grating lobes ($d = \lambda/4, \beta_x = \beta_y = 0$):

Fig. 6.24, pp.313 Balanis
3-D pattern of a 5-element square planar uniform array without grating lobes ($d = \lambda / 2, \beta_x = \beta_y = 0$):

Notice the considerable decrease in the beamwidth as the spacing is increased from $\lambda / 4$ to $\lambda / 2$. 

Fig. 6.25, pp.314, Balanis
1.2 The beamwidth of a planar array

A simple procedure, proposed by R.S. Elliot\(^1\) will be outlined. It is based on the use of the beamwidths of the linear arrays building the planar array.

For a large array, whose maximum is near the broad side, the elevation plane HPBW is approximately:

\[
\theta_h = \sqrt{\frac{1}{\cos^2 \theta_0 \left[ \Delta \theta_x^{-2} \cos^2 \phi_0 + \Delta \theta_y^{-2} \sin^2 \phi_0 \right]}} \quad (18.18)
\]

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\(^1\)"Beamwidth and directivity of large scanning arrays", *The Microwave Journal*, Jan. 1964, pp.74-82
where: \((\theta_0, \phi_0)\) specify the main-beam direction;

\(\Delta \theta_x\) is the HPBW of a linear broadside array whose number of elements \(M\) and amplitude distribution is the same as that of the \(x\)-axis linear arrays building the planar array;

\(\Delta \theta_y\) is the HPBW of a linear BSA whose number of elements \(N\) and amplitude distribution is the same as those of the \(y\)-axis linear arrays building the planar array.

The HPBW in the plane, which is orthogonal to the \(\phi = \phi_0\) plane and contains the maximum, is:

\[
\phi_h = \sqrt{\frac{1}{\Delta \theta_x^{-2} \sin^2 \phi_0 + \Delta \theta_y^{-2} \cos^2 \phi_0}}
\] (18.19)

For a square array \((M = N)\) with amplitude distributions along the \(x\) and \(y\) axes of the same type, equations (18.18) and (18.19) reduce to:

\[
\theta_h = \frac{\Delta \theta_x}{\cos \theta_0} = \frac{\Delta \theta_y}{\cos \theta_0}
\] (18.20)

\[
\phi_h = \Delta \theta_x = \Delta \theta_y
\] (18.21)

From (18.20), it is obvious that the HPBW in the elevation plane very much depends on the elevation angle \(\theta_0\) of the main beam. The HPBW in the azimuthal plane \(\phi_h\) does not depend on the elevation angle \(\theta_0\).

The beam solid angle of the planar array can be approximated by:

\[
\Omega_A = \theta_h \phi_h
\] (18.22)

or
1.3 Directivity

The general expression for the calculation of the directivity of an array is:

\[
\Omega_A = \frac{\Delta \theta_x \Delta \theta_y}{\cos^2 \theta_0 \sqrt{\left[ \sin^2 \phi_0 + \frac{\Delta \theta_y^2}{\Delta \theta_x^2} \cos^2 \phi_0 \right] \left[ \sin^2 \phi_0 + \frac{\Delta \theta_x^2}{\Delta \theta_y^2} \cos^2 \phi_0 \right]}}
\] (18.23)

\[
D_0 = 4\pi \frac{|AF(\theta_0, \phi_0)|^2}{\int_0^{2\pi} \int_0^{\pi} |AF(\theta_0, \phi_0)|^2 \sin \theta d\theta d\phi}
\] (18.24)

For large planar arrays, which are nearly broadside, (18.24) reduces to:

\[
D_0 = \pi D_x D_y \cos \theta_0
\] (18.25)

where \( D_x \) is the directivity of the respective linear BSA, \( x \)-axis; \( D_y \) is the directivity of the respective linear BSA, \( y \)-axis.

One can also use the array solid beam angle \( \Omega_A \) in (18.23) to calculate the approximate directivity of a nearly broadside planar array:

\[
D_0 \approx \frac{\pi^2}{\Omega_A[\text{sr}]} \approx \frac{32400}{\Omega_A[\text{deg}^2]}
\] (18.26)

Remember:
1) The main beam direction is controlled through the phase shifts, \( \beta_x \) and \( \beta_y \).
2) The beamwidth and side-lobe levels are controlled through the amplitude distribution.
2. Circular array

2.1 Array factor

The normalized field can be written as:

\[ E(r, \theta, \phi) = \sum_{n=1}^{N} a_n \frac{e^{-jkR_n}}{R_n} \]  

(18.27)

where:

\[ R_n = \sqrt{r^2 + a^2 - 2ar \cos \psi_n} \]  

(18.28)

For \( r \gg a \), (18.28) reduces to:

\[ R_n = r - a \cos \psi_n \approx r - a(\hat{a}_{\rho_n} \cdot \hat{r}) \]  

(18.29)

In rectangular coordinate system:

\[
\hat{a}_{\rho_n} = \hat{x} \cos \phi_n + \hat{y} \sin \phi_n \\
\hat{r} = \hat{x} \sin \theta \cos \phi + \hat{y} \sin \theta \sin \phi + \hat{z} \cos \theta
\]
Therefore:
\[ R_n = r - a \sin \theta (\cos \phi_n \cos \phi + \sin \phi_n \sin \phi) \quad (18.30) \]

Finally, \( R_n \) is approximated in the phase terms as:
\[ R_n = r - a \sin \theta \cos (\phi - \phi_n) \quad (18.31) \]

For the amplitude term, the approximation
\[ \frac{1}{R_n} \approx \frac{1}{r}, \text{ all } n \quad (18.32) \]

is made.

Assuming the approximations (18.31) and (18.32) are valid, the far-zone array field is reduced to:
\[
E(r, \theta, \phi) = \frac{e^{-jkr}}{r} \sum_{n=1}^{N} a_n e^{jka \sin \theta \cos(\phi - \phi_n)} \quad (18.33)
\]

where: \( a_n \) is the excitation coefficient (amplitude and phase);
\[ \phi_n = \frac{2\pi}{N} n \quad \text{is the angular position of the } n\text{-th element.} \]

In general, the excitation coefficient can be represented as:
\[ a = I_n e^{j\alpha_n}, \quad (18.34) \]

where \( I_n \) is the amplitude term, and \( \alpha_n \) is the phase of the excitation of the \( n \)-th element relative to a chosen array element of zero phase.

\[ \Rightarrow E(r, \theta, \phi) = \frac{e^{-jkr}}{r} \sum_{n=1}^{N} I_n e^{j[ka \sin \theta \cos(\phi - \phi_n) + \alpha_n]} \quad (18.35) \]

The AF is obtained as:
\[ \text{AF}(\theta, \phi) = \sum_{n=1}^{N} I_n e^{j[ka \sin \theta \cos(\phi - \phi_n) + \alpha_n]} \quad (18.36) \]

Expression (18.36) represents the AF of a circular array of \( N \) equispaced elements. The maximum of the AF occurs when all the phase terms in (18.36) equal unity, or:
\[ ka \sin \theta \cos(\phi - \phi_n) + \alpha_n = 2m\pi, \quad m = 0, \pm 1, \pm 2, \text{ all } n \quad (18.37) \]
The principal maximum \((m = 0)\) is defined by the direction 
\((\theta_0, \phi_0)\), for which:
\[
\alpha_n = -ka \sin \theta_0 \cos (\phi_0 - \phi_n), \quad n = 1, 2, ..., N \tag{18.38}
\]
If a circular array is required to have maximum radiation in the direction \((\theta_0, \phi_0)\), then the phases of its excitations will have to
fulfil (18.38). The AF of such an array is:
\[
AF(\theta, \phi) = \sum_{n=1}^{N} I_n e^{jka[\sin \theta \cos (\phi - \phi_n) - \sin \theta_0 \cos (\phi_0 - \phi_n)]} \tag{18.39}
\]
\[
AF(\theta, \phi) = \sum_{n=1}^{N} I_n e^{jka(\cos \psi_n - \cos \psi_{0n})} \tag{18.40}
\]
Here:
\[
\psi_n = \cos^{-1} [\sin \theta \cos (\phi - \phi_n)] \quad \text{is the angle between } \hat{r} \text{ and } \hat{a}_n ;
\]
\[
\psi_{0n} = \cos^{-1} [\sin \theta_0 \cos (\phi_0 - \phi_n)] \quad \text{is the angle between } \hat{a}_n \text{ and } \hat{r}_{\text{max}} \text{ pointing in the direction of maximum radiation.}
\]

As the radius of the array \(a\) becomes very large as compared to \(\lambda\), the directivity of the uniform circular array \((I_n = I_0, \quad \text{all } n)\)
approaches the value of \(N\).
Uniform circular array 3-D pattern ($N=10$, $ka = \frac{2\pi}{\lambda}$, $a = 10$)