LECTURE 2: Introduction into the Theory of Radiation

(*Maxwell's equations – revision. Power density and Poynting vector – revision. Radiated power – definition. Basic principle of radiation. Vector and scalar potentials – revision. Far fields and vector potentials.*)

1. Maxwell's equations – revision.

(a) the law of induction (Faraday's law):

$$-\nabla\times\vec{E} = \frac{\partial\vec{B}}{\partial t} + \vec{M}^{\clubsuit} \tag{2.1}$$

$$\oint_c \vec{E}d\vec{c} = -\frac{\partial}{\partial t}\iint_{S_{[c]}} \vec{B}d\vec{s} \quad \Leftrightarrow \quad e = -\frac{\partial\Psi}{\partial t} \tag{2.1-i}$$

$\vec{E}$ (V/m) electric field (electric field intensity)
$\vec{B}$ (T=Wb/m^2) magnetic flux density
$\vec{M}$ (V/m^2) magnetic current density♣
Ψ (Wb=V·s) magnetic flux
e (V) electromotive force

(b) Ampere's law, generalized by Maxwell to include the displacement current $\partial\vec{D}/\partial t$:

$$\nabla\times\vec{H} = \frac{\partial\vec{D}}{\partial t} + \vec{J} \tag{2.2}$$

$$\oint_c \vec{H}d\vec{c} = \iint_{S_{[c]}}\left(\frac{\partial\vec{D}}{\partial t} + \vec{J}\right)d\vec{s} \quad \Leftrightarrow \quad I = \oint_c \vec{H}d\vec{c} \tag{2.2-i}$$

$\vec{H}$ (A/m) magnetic field (magnetic field intensity)
$\vec{D}$ (C/m^2) electric flux density (electric displacement)
$\vec{J}$ (A/m^2) electric current density
I (A) electric current

♣ $\vec{M}$ is a fictitious quantity, which renders Maxwell's equations symmetrical and which proves a very useful mathematical tool when solving EM boundary value problems applying equivalence theorem.

(c) Gauss' electric law:

$$\nabla \cdot \vec{D} = \rho \tag{2.3}$$

$$\oiint_S \vec{D} d\vec{s} = \iiint_{V_{[s]}} \rho dv = Q \tag{2.3-i}$$

ρ (C/m^3) electric charge density

Q (C) electric charge

Equation (2.3) follows from equation (2.2) and the continuity relation:

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \tag{2.4}$$

Hint: Take the divergence of both sides of (2.2).

(d) Gauss' magnetic law:

$$\nabla \cdot \vec{B} = \rho_m \text{**} \tag{2.5}$$

The equation $\nabla \cdot \vec{B} = 0$ follows from equation (2.1), provided that $\vec{M} = 0$.

Maxwell's equations alone are insufficient to solve for the four vector quantities: $\vec{E}, \vec{D}, \vec{H}$ and $\vec{B}$ (twelve scalar quantities). Two additional vector equations are needed.

(e) Constitutive relationships

The constitutive relationships describe the properties of matter with respect to electric and magnetic forces.

$$\vec{D} = \vec{\vec{\varepsilon}} \cdot \vec{E} \tag{2.6}$$

$$\vec{B} = \vec{\vec{\mu}} \cdot \vec{H} \tag{2.7}$$

In the most general case of anisotropic medium, the dielectric permittivity and the magnetic permeability are *tensors*. In vacuum, which is isotropic, the dielectric permittivity and the magnetic permeability are constants (or tensors whose *diagonal elements only* are non-zero and are the same): $\varepsilon_0 = 8.854187817 \times 10^{-12}$ F/m, $\mu_0 = 4\pi \times 10^{-7}$ H/m. In isotropic medium, the vectors $\vec{D}$ and $\vec{E}$ are collinear, and so are the vectors $\vec{B}$ and $\vec{H}$.

** ρ_m is a fictitious quantity introduced via the continuity relation $\nabla \cdot \vec{M} = -\partial \rho_m / \partial t$. In nature, $\nabla \cdot \vec{B} = 0$.

Dielectric properties relate to the electric field (electric force). Dielectric materials with relative dielectric permittivity (dielectric constant) ε_r > 1 are built of atomic/molecular sub-domains, which have the properties of dipoles. In external electric field, the dipoles tend to orient in such a way that their own fields have a cancellation effect on the external field. The electric force $\vec{F}_e = Q\vec{E}$ exerted on a point charge Q from a source Q_s in such medium will be ε_r times weaker than the electric force of the same source in vacuum.

On the contrary, magnetic materials with relative permeability (magnetic constant) μ_r > 1 are made of sub-domains, which tend to orient in external magnetic field in such a way, that their own magnetic fields align with the external field. The magnetic force $\vec{F}_m = Q\vec{v}\times\vec{B}$ exerted on a moving point charge Q in such a medium will be μ_r times stronger than the force that this same source (e.g. electric currents) would create in vacuum.

We shall be mostly concerned with isotropic media, i.e. media where the equations $\vec{B} = \mu_0\mu_r\vec{H}$ and $\vec{D} = \varepsilon_0\varepsilon_r\vec{E}$ hold.

(f) Time-harmonic field analysis

In harmonic analysis of EM fields, the field phasors are introduced:

$$\begin{aligned} \vec{e}(x,y,z,t) &= \mathrm{Re}\left\{\vec{E}(x,y,z)e^{j\omega t}\right\} \\ \vec{h}(x,y,z,t) &= \mathrm{Re}\left\{\vec{H}(x,y,z)e^{j\omega t}\right\} \end{aligned} \qquad (2.8)$$

For clarity, from this point on, we shall denote time dependent field vectors with lower-case letters, while their phasors will be denoted with upper-case letters. Complex-conjugate quantities will be denoted with the * sign.

The phasor equations are obtained from the time dependent equations by simple substitution of the following correspondences:

$$f(x,y,z,t) \doteq F(x,y,z)$$

$$\frac{\partial f_{(x,y,z,t)}}{\partial t} \doteq j\omega F(x,y,z)$$

$$\frac{\partial f}{\partial \xi} \doteq \frac{\partial F}{\partial \xi} \quad , \quad \xi = x,y,z$$

For example, Maxwell's equations in phasor form are obtained as:

$$\nabla\times\vec{H} = j\omega(\varepsilon' - j\varepsilon'')\vec{E} + \sigma\vec{E} + \vec{J} = j\omega\bar{\varepsilon}\vec{E} + \vec{J}, \quad \bar{\varepsilon} = \varepsilon' - j\left(\varepsilon'' + \frac{\sigma}{\omega}\right) \qquad (2.9)$$

$$-\nabla\times\vec{E} = j\omega(\mu' - j\mu'')\vec{H} + \vec{M} = j\omega\bar{\mu}\vec{H} + \vec{M} \tag{2.10}$$

This is the general form of Maxwell's equations. They include the equivalent (fictitious) magnetic currents $\vec{M}$. The dielectric losses (due to alternate field conductivity $\omega\varepsilon''$) and the static field conductivity losses σ are both represented by the imaginary part of the *complex dielectric permittivity* $\bar{\varepsilon}$. Often, the dielectric losses are represented by the dielectric loss-angle δ_d:

$$\bar{\varepsilon} = \varepsilon'\left[1 - j\left(\frac{\varepsilon''}{\varepsilon'} + \frac{\sigma}{\omega\varepsilon'}\right)\right] = \varepsilon'\left[1 - j\left(\tan\delta_d + \frac{\sigma}{\omega\varepsilon'}\right)\right]$$

Similarly, the magnetic losses are described by the imaginary part of the *complex magnetic permeability* $\bar{\mu}$ or by the magnetic loss-angle δ_m:

$$\bar{\mu} = \mu' - j\mu'' = \mu'\left(1 - j\frac{\mu''}{\mu'}\right) = \mu'(1 - j\tan\delta_m)$$

In antenna theory, we are mostly concerned with *isotropic*, *homogeneous* and *loss-free* regions.

2. Power density and vector of Poynting - revision. Radiated power.

2.1. Poynting vector – revision.

In time-domain analysis, the Poynting vector is defined as

$$\vec{p}(t) = \vec{e}(t)\times\vec{h}(t),\ \text{W/m}^2 \tag{2.11}$$

As follows from Poynting's theorem, $\vec{p}$ is a vector representing the density and the direction of the EM power flow. Thus, the total power leaving certain volume V is obtained as:

$$\Pi(t) = \oint\!\!\!\oint_{S_{[V]}} \vec{p}_{(t)} d\vec{s},\ \text{W} \tag{2.12}$$

Since

$$e(t) = \text{Re}\left\{\vec{E}e^{j\omega t}\right\} = \frac{1}{2}\left(\vec{E}e^{j\omega t} + \vec{E}^{*}e^{-j\omega t}\right), \tag{2.13}$$

and

$$h(t) = \text{Re}\left\{\vec{H}e^{j\omega t}\right\} = \frac{1}{2}\left(\vec{H}e^{j\omega t} + \vec{H}^{*}e^{-j\omega t}\right), \tag{2.14}$$

it can be derived that

$$\vec{p}(t) = \underbrace{\frac{1}{2}\mathrm{Re}\left\{\vec{E}\times\vec{H}^*\right\}}_{\vec{p}_{av}} + \frac{1}{2}\mathrm{Re}\left\{\vec{E}\times\vec{H}\cdot e^{2\cdot j\omega t}\right\} \quad (2.15)$$

The first term in (2.15) has no time dependence. It is the average value, around which the power flux density fluctuates. It is a vector of unchanging direction showing a constant outflow (positive value) or inflow (negative value) of EM power. In terms of circuit theory, it describes the active power flow, which is the time-average power flux:

$$\Pi_{av} = \oiint_{S_{[V]}} \vec{p}_{av} d\vec{s} \quad (2.16)$$

The second term in (2.15) is a vector changing its direction with a double frequency (2ω). It describes the reactive power flow, i.e. the power, which fluctuates in space (propagates to and fro) without contribution to the overall transport of energy in any direction.

Definition: The complex Poynting vector is the vector

$$\vec{P} = \frac{1}{2}\vec{E}\times\vec{H}^*, \quad (2.17)$$

whose real part is equal to the average power flux density.

2.2. Radiated power.

Definition: Radiated power is the average power radiated by the antenna:

$$\Pi_{rad} = \oiint_{S_{[V]}} \vec{p}_{av} d\vec{s} = \oiint_{S_{[V]}} \mathrm{Re}\left\{\vec{P}\right\} d\vec{s} = \frac{1}{2}\oiint_{S_{[V]}} \mathrm{Re}\left\{\vec{E}\times\vec{H}^*\right\} d\vec{s} \quad (2.18)$$

3. Basic principle of radiation.

Radiation is produced by accelerated or decelerated charge (time-varying current element)

Definition: A current element ($I\Delta l$) is a filament of length Δl and current I.

The concept of current element is essential in the theory of EM wave radiation, since the time-varying current element is the elementary source of EM radiation. It has the same significance as the concept of a point charge in electrostatics. The field radiated by a complex antenna in a linear medium can be analyzed by making use of the superposition principle after decomposing the antenna into elementary sources (i.e. into current elements).

Assume the existence of a piece of very thin wire, where electric currents can be excited. The current i flowing through the wire cross-section $\vartriangle S$ is defined as the amount of charge passing through $\vartriangle S$ in 1 second:

$$i = \rho \cdot \vartriangle S \cdot \vartriangle l_1 = \rho \cdot \vartriangle S \cdot v, \quad \text{A} \tag{2.19}$$

where:

ρ (C/m^3) is the electric charge volume density

v (m/s) is the velocity of the charges normal to the cross-section

$\vartriangle l_1$ (m/s) is the distance traveled by a charge in 1 second

Equation (2.19) can be also written as

$$\vec{j} = \rho \cdot \vec{v}, \quad \text{A/m}^2 \tag{2.20}$$

where $\vec{j}$ is the electric current density. The product $\lambda = \rho \cdot \vartriangle S$ is the charge per unit length (charge line density) along the wire. Thus, from (2.19) it follows that

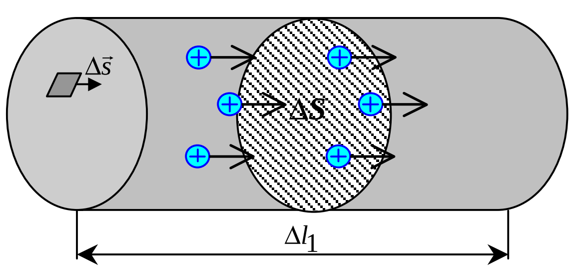
$$i = v \cdot \lambda \tag{2.21}$$

It is then obvious that

$$\frac{di}{dt} = \lambda \frac{dv}{dt} = \lambda a, \tag{2.22}$$

where a (m/s^2) is the acceleration of the charge. A time-varying current source would then be proportional to the amount of charge q enclosed in the volume of the current element and to its acceleration:

$$\vartriangle l \frac{di}{dt} = \vartriangle l \cdot \lambda \cdot a = q \cdot a \tag{2.23}$$



It is not immediately obvious from Maxwell's equations that the time-varying current is the source of propagating EM field. The system of the two first-order Maxwell's equations in isotropic medium, though, can be easily reduced to a single second-order equation either for the $\vec{E}$ vector, or for the $\vec{H}$ vector.

$$\left| \begin{aligned} -\nabla\times\vec{e} &= \mu\frac{\partial\vec{h}}{\partial t} \\ \nabla\times\vec{h} &= \varepsilon\frac{\partial\vec{e}}{\partial t}+\vec{j} \end{aligned} \right. \tag{2.24}$$

By taking the curl of both sides of the first equation in (2.24) and by making use of the second equation in (2.24), one obtains:

$$\nabla\times\nabla\times\vec{e}+\mu\varepsilon\frac{\partial^2\vec{e}}{\partial t^2}=-\mu\frac{\partial\vec{j}}{\partial t} \tag{2.25}$$

From equation (2.25), it is obvious that the time derivative of the electric currents is the source for the wave-like propagation of the vector $\vec{e}$ in homogeneous and isotropic medium. In an analogous way, one can obtain the wave equation for the magnetic field $\vec{H}$ and its sources:

$$\nabla\times\nabla\times\vec{h}+\mu\varepsilon\frac{\partial^2\vec{h}}{\partial t^2}=\nabla\times\vec{j} \tag{2.26}$$

To create charge acceleration/deceleration one needs sources of electromotive force and/or discontinuities of the medium in which the charges move. Such discontinuities can be bends or open ends of wires, change in the electrical properties of the region, etc. There is the summary of the causes for radiation:

- If charge is not moving, current is zero $\Rightarrow$ no radiation
- If charge is moving with a uniform velocity $\Rightarrow$ no radiation
- If charge is accelerated due to electromotive force or is decelerated due to discontinuities, such as termination, bend, curvature $\Rightarrow$ radiation occurs

4. Vector and scalar potentials – review.

With very few exceptions, antennas are assumed to radiate in open (free) space, which determines the specifics of the arising EM problems. Often, the EM sources (currents and charges) are more or less accurately known. These sources are then assumed to radiate (in unbounded free space) and it is required to determine the resulting EM field. Such problems, where the sources are known, and the reaction (result) is to be determined are called *analysis* problems (direct problems). The inverse (*design*) problem of finding the sources of a known result (reaction) are much more difficult and we shall not consider them here. To ensure the uniqueness of the solution in an open (unbounded) problem, one has to impose the radiation boundary condition (BC) on the EM field vectors, i.e. for distances far away from the source ($r \to \infty$)

$$\begin{vmatrix} r(\vec{E} - \eta \vec{H} \times \hat{r}) \to 0 \\ r(\vec{H} - \frac{1}{\eta} \hat{r} \times \vec{E}) \to 0 \end{vmatrix} \tag{2.27}$$

The above BCs are also known as the Sommerfeld radiation BCs. Here, η is the intrinsic impedance of the medium.

The specifics of the antenna problems lead to the introduction of auxiliary vector potential functions, which allow simpler and compact solutions.

It is customary to perform the EM analysis for the case of time-harmonic fields, i.e. in terms of phasors. This course will adhere to the tradition. Therefore, from now on, all field quantities (vectors and scalars) are to be understood as ***complex phasor quantities, whose magnitudes correspond to the magnitudes of the respective sine waves***.

4.1. The magnetic vector potential $\vec{A}$

We shall first consider only electric sources ($\vec{J}$ and ρ), which are actual currents and charges.

$$\begin{vmatrix} \nabla \times \vec{E} = -j\omega\mu\vec{H} \\ \nabla \times \vec{H} = j\omega\varepsilon\vec{E} + \vec{J} \end{vmatrix} \tag{2.28}$$

Since $\nabla \cdot \vec{B} = 0$, one can assume that

$$\vec{B} = \nabla \times \vec{A} \tag{2.29}$$

Substituting (2.29) in (2.28) yields:

$$\begin{vmatrix} \vec{E} = -j\omega\vec{A} - \nabla\Phi \\ j\omega\varepsilon\vec{E} = \nabla \times \left(\frac{1}{\mu} \nabla \times \vec{A} \right) - \vec{J} \end{vmatrix} \tag{2.30}$$

From (2.30) it is obvious that a single equation can be written for $\vec{A}$. In isotropic, homogeneous region, this equation is obtained as:

$$\nabla \times \nabla \times \vec{A} + j\omega\mu\varepsilon(j\omega\vec{A} + \nabla\Phi) = \mu\vec{J} \tag{2.31}$$

Here, Φ denotes the electric scalar potential, which plays essential role in the analysis of electrostatic fields. To uniquely define the vector field $\vec{A}$, we need to define not only its curl, but also its divergence. There are no restrictions in defining $\nabla \cdot \vec{A}$. Since $\nabla \times \nabla \times = \nabla\nabla \cdot - \nabla^2$, equation (2.31) can be simplified by assuming that

$$\nabla \cdot \vec{A} = -j\omega\mu\varepsilon\Phi \tag{2.32}$$

Equation (2.32) is known as the Lorentz' gauge condition. It reduces (2.31) to

$$\nabla^2 \vec{A} + \omega^2 \mu\varepsilon \vec{A} = -\mu \vec{J} \tag{2.33}$$

If the region is lossless, then μ and ε are real numbers, and (2.33) can be written as:

$$\nabla^2 \vec{A} + \beta^2 \vec{A} = -\mu \vec{J}\,, \tag{2.34}$$

where $\beta = \omega\sqrt{\mu\varepsilon}$ is the phase constant. If the region is lossy (which is rarely the case in antenna problems), complex permittivity $\bar{\varepsilon}$ and complex permeability $\bar{\mu}$ are introduced. Then, (2.33) can be written in the form:

$$\nabla^2 \vec{A} - \gamma^2 \vec{A} = -\mu \vec{J} \tag{2.35}$$

Here, $\gamma = \alpha + j\beta = j\omega\sqrt{\bar{\mu}\bar{\varepsilon}}$ is the propagation constant, and α is the attenuation constant. For example, if the region has losses due to non-zero conductivity σ, the complex dielectric permittivity is introduced as:

$$\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon) = -\omega^2\mu \underbrace{\left(\varepsilon + \frac{\sigma}{j\omega}\right)}_{\bar{\varepsilon}} \tag{2.36}$$

4.2. The electric vector potential $\vec{F}$.

The magnetic field is a solenoidal field, i.e. $\nabla \cdot \vec{B} = 0$, because there are no physically existing magnetic charges. Therefore, there are no physically existing magnetic currents either. However, the fictitious (equivalent) magnetic currents $\vec{M}$ are a very useful tool when applied with the equivalence principle. These currents are introduced in Maxwell's equations in a manner dual to that of the electric currents $\vec{J}$. Now, we shall consider the field due to magnetic sources *only*, i.e. we assume that $\vec{J} = 0$ and that $\rho = 0$, and therefore $\nabla \cdot \vec{D} = 0$. Then, the system of Maxwell's equations is:

$$\left|\begin{array}{l} \nabla \times \vec{E} = -j\omega\mu\vec{H} - \vec{M} \\ \nabla \times \vec{H} = j\omega\varepsilon\vec{E} \end{array}\right. \tag{2.37}$$

Since $\vec{D}$ is solenoidal, it can be expressed as the curl of a vector, namely the electric vector potential $\vec{F}$:

$$\vec{D} = -\nabla \times \vec{F} \tag{2.38}$$

Equation (2.38) is substituted in the system (2.37). All mathematical transformations are analogous to those made in Section 4.1. Finally, it is shown that a field due to magnetic sources is entirely described by a single vector $\vec{F}$, which satisfies the Helmholtz' equation

$$\nabla^2 \vec{F} + \omega^2 \mu\varepsilon \vec{F} = -\varepsilon \vec{M}\,, \tag{2.39}$$

provided that the Lorentz' gauge is imposed in the form

$$\nabla \cdot \vec{F} = -\,j\omega\mu\varepsilon\Psi \tag{2.40}$$

Here, Ψ represents the magnetic scalar potential.

In linear medium, a field due to both types of sources (magnetic and electric) can be solved for by superimposing the partial field due to the electric sources only and the one due to magnetic sources only.

TABLE 2.1: FIELD VECTORS IN TERMS OF VECTOR-POTENTIALS

Magnetic vector-potential $\vec{A}$ (electric sources only)	Electric vector-potential $\vec{F}$ (magnetic sources only)
$\vec{B} = \nabla \times \vec{A},\ \vec{H} = \frac{1}{\mu}\nabla \times \vec{A}$ $\vec{E} = -\,j\omega\vec{A} - \frac{j}{\omega\mu\varepsilon}\nabla\nabla \cdot \vec{A}$ or $\vec{E} = \frac{1}{j\omega\mu\varepsilon}\nabla \times \nabla \times \vec{A} - \frac{\vec{J}}{j\omega\varepsilon}$	$\vec{D} = -\nabla \times \vec{F},\ \vec{E} = -\frac{1}{\varepsilon}\nabla \times \vec{F}$ $\vec{H} = -\,j\omega\vec{F} - \frac{j}{\omega\mu\varepsilon}\nabla\nabla \cdot \vec{F}$ or $\vec{H} = \frac{1}{j\omega\mu\varepsilon}\nabla \times \nabla \times \vec{F} - \frac{\vec{M}}{j\omega\mu}$

5. Retarded potentials - review.

Retarded potential is a term usually used to denote the solution of the inhomogeneous Helmholtz' equation (in the frequency domain) or that of the inhomogeneous wave equation (in the time domain) in an unbounded region.

Assume that an infinitesimal current source (in the form of a Dirac δ-function) exists at the origin of the coordinate system, and that it has a current density vector with a z-component only, i.e.

$$d\vec{J} = \hat{z}\ J_z \delta(x)\delta(y)\delta(z) \tag{2.41}$$

Then, according to (2.34), the magnetic vector potential $\vec{A}$ will also have only a z-component governed by the following equation in lossless medium:

$$\nabla^2 A_z + \beta^2 A_z = -\mu J_z \qquad (2.42)$$

The field A_z has a spherical symmetry (no dependence on the observation angles θ and φ) as implied by the spherical symmetry of a point source. Thus, equation (2.42) reduces to an ordinary differential equation (ODE) with derivatives only with respect to the distance *r*, when one writes the Laplace operator $\nabla^2 \equiv \Delta$ in spherical coordinates:

$$\frac{d^2 A_z}{dr^2} + \frac{2}{r}\frac{dA_z}{dr} + \beta^2 A_z = -\mu J_z \qquad (2.43)$$

Except at the source, the field A_z satisfies the homogeneous version of (2.43):

$$\frac{d^2 A_z}{dr^2} + \frac{2}{r}\frac{dA_z}{dr} + \beta^2 A_z = 0 \qquad (2.44)$$

The solutions of (2.44) are well known in the case of an unbounded region:

$$A_{z_1} = C_1 \frac{e^{-j\beta r}}{r} \qquad (2.45)$$

$$A_{z_2} = C_2 \frac{e^{+j\beta r}}{r} \qquad (2.46)$$

The solution (2.46) represents an incoming wave, which cannot be a contribution of the given source. It is discarded. The solution (2.45) represents an outgoing wave and is physically feasible. This solution is called a retarded potential, which refers to the finite velocity with which the field disturbances (waves) travel and the finite time interval they need in order to reach certain point of observation. The constant C_1 has to be determined, making use of the source and the boundary conditions. Since the region is unbounded, the only BC left is the scalar radiation BC♣, which is already satisfied by (2.45). By integrating (2.43) inside a spherical volume surrounding the source (see Appendix I), one obtains $C_1 = (1/4\pi)\mu J_z$. Therefore, the elementary field produced by an infinitesimal current source, is described only by the *z*-component of the magnetic vector potential, which is:

$$dA_z = \mu J_z \frac{e^{-j\beta r}}{4\pi r} \qquad (2.47)$$

If the point source is not at the origin, but is at some point, *Q*, of a radius-vector $\vec{r}\,'$, then the variable *r* in (2.50) must be substituted by *R*, where *R* is the

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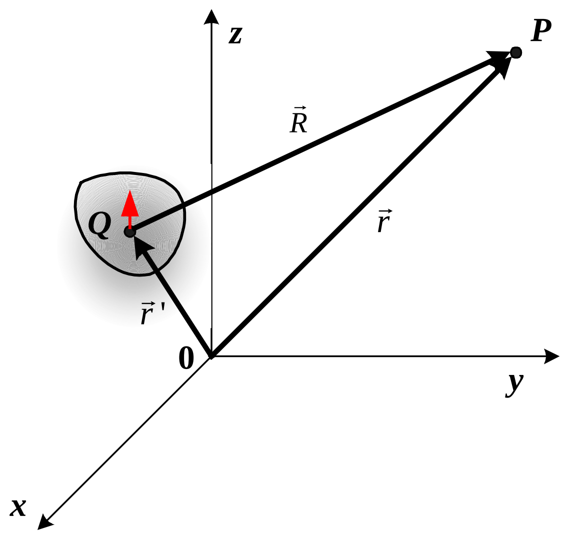
♣ The radiation BC for a scalar function Φ satisfying the wave equation states that $\lim_{r\to\infty} r \cdot \left(\frac{\partial \Phi}{\partial r} + j\beta\Phi \right) = 0$

distance between the source at $Q(\vec{r}\,')$ and the observation point $P(\vec{r}\,)$: $R = |\vec{r} - \vec{r}\,'|$. Explicitly:

$$dA_z = \mu J_z \frac{e^{-j\beta R}}{4\pi R} \tag{2.48}$$



Let us consider the case of a continuously distributed source in the form of $J_z(x', y', z')$. It can be viewed as a cluster of point sources whose joint retarded potential will produce the overall A_z potential in linear medium (principle of superposition):

$$A_z(P) = \iiint_V dA_z = \iiint_V \mu J_z(Q) \frac{e^{-j\beta R}}{4\pi R} dv_Q \tag{2.49}$$

To further generalize the above formula, one should assume the existence of source currents of arbitrary directions, which would produce partial magnetic vector potentials in any directions. Note that a current element in the $\hat{\xi}$ direction will result in a vector potential $\vec{A} = A_\xi \hat{\xi}$ in the same direction (unless the medium is anisotropic). Thus,

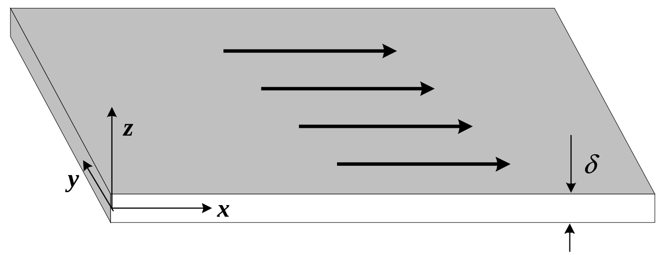
$$\vec{A}(P) = \iiint_V \mu \vec{J}(Q) \frac{e^{-j\beta R}}{4\pi R} dv_Q \tag{2.50}$$

The solution for the electric vector potential due to magnetic current sources $\vec{M}(Q)$ is analogous:

$$\vec{F}(P) = \iiint_V \varepsilon \vec{M}(Q) \frac{e^{-j\beta R}}{4\pi R} dv_Q \tag{2.51}$$

Finally, one should recall that not only *volume* sources are used to model the currents of a radiator. A useful approximation, especially of currents at conducting boundaries, is the *surface* current density (or simply surface currents):

$$\vec{J}_s(x,y) = \lim_{\delta \to 0} \int_{-\delta/2}^{\delta/2} \vec{J}(x,y,z)dz, \text{ A/m}$$

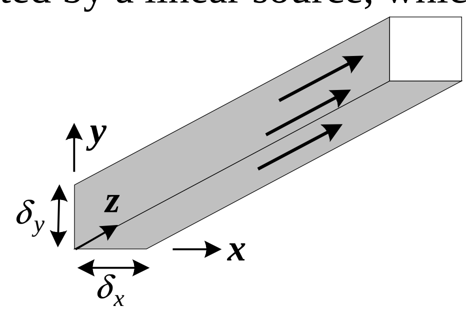


The magnetic vector potential $\vec{A}$ produced by distributed surface currents will then be expressed as:

$$\vec{A}(P) = \iint_S \mu \vec{J}_s(Q) \frac{e^{-j\beta R}}{4\pi R} ds_Q \tag{2.52}$$

Currents on a very thin wire are nicely approximated by a linear source, which is exactly the current *I* flowing through the wire:

$$\vec{I}(z) = \lim_{\substack{\delta_x \to 0 \\ \delta_y \to 0}} \iint_{\delta_x \delta_y} \vec{J}(x,y,z)dxdy, \text{ A}$$



The respective potential is:

$$\vec{A}(P) = \int_L \mu \vec{I}(Q) \frac{e^{-j\beta R}}{4\pi R} d\vec{l}_Q \tag{2.53}$$

6. Far fields and vector potentials.

Antennas are sources of finite physical dimensions. The further away from the antenna the observation point is, the more the wave looks like a spherical wave and the more the antenna looks like a point source regardless of its actual shape. In such cases, we talk about *far fields* and *far zone*. The exact meaning of these terms will be discussed later. For now, we will simply assume that the vector potentials behave like spherical waves, when the observation point is far from the source:

$$\vec{A} \simeq \left[\hat{R} \cdot A_R(\theta,\varphi) + \hat{\theta} \cdot A_\theta(\theta,\varphi) + \hat{\varphi} \cdot A_\varphi(\theta,\varphi) \right] \frac{e^{-jkR}}{R}, \quad R \to \infty \tag{2.54}$$

Here, $(\hat{R},\hat{\theta},\hat{\varphi})$ are the unit vectors of the spherical coordinate system centred on the antenna. The term e^{-jkR} shows propagation along $\hat{R}$ away from the antenna with the speed of light. The term $1/R$ shows the spherical spread of the potential in space, which results in a decrease of its magnitude as the radius of the sphere increases.

Notice an important feature of the far-field potential: the dependence on the distance R is separable from the dependence on the observation angle (θ,φ), and it is the same for any antenna: e^{-jkR}/R.

Formula (2.54) is a *far-field approximation* of the vector potential at distant points. One can arrive at it starting from the original integral in (2.50). When the observation point is very far from the source volume, the distance R_{PQ} between the observation and the integration points varies only slightly as Q sweeps the volume. It is almost the same as the distance from the origin to the observation point R since we usually centre the coordinate system close to the sources. The following first-order approximation is made for the integrand:

$$\frac{e^{-jkR_{PQ}}}{R_{PQ}} \approx \frac{e^{-jk(R-\hat{R}\cdot\vec{R}')}}{R} \tag{2.55}$$

Here,

$\vec{R}$ is the position vector of the observation point P, and $R = |\vec{R}|$;

$\vec{R}'$ is the position vector of the integration point Q.

Equation (2.55) is called the *far-field approximation*. It is illustrated in the figures below. The first figure shows the real problem. The second one shows the approximated problem, where it is in effect assumed that the vectors $\vec{R}_{PQ}$ and $\vec{R}$ are parallel.

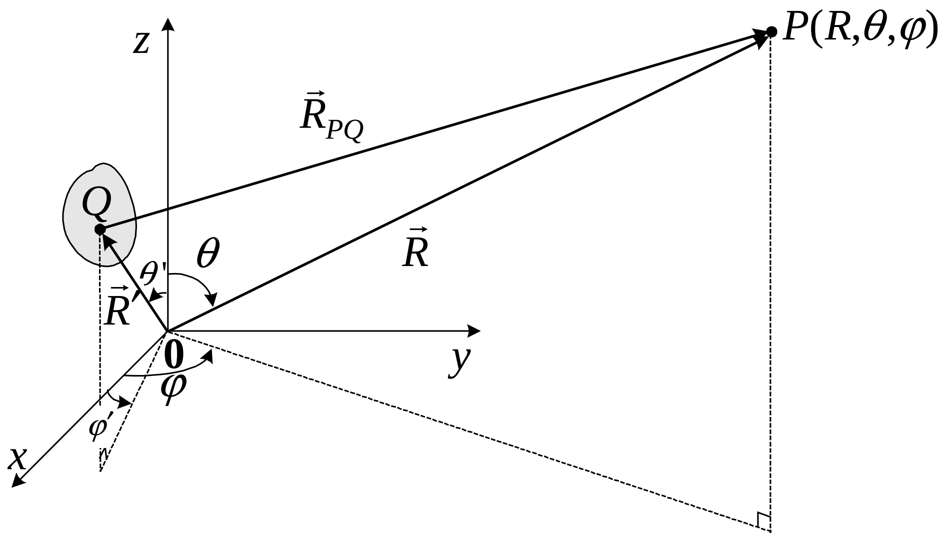


Figure 1: Original problem.

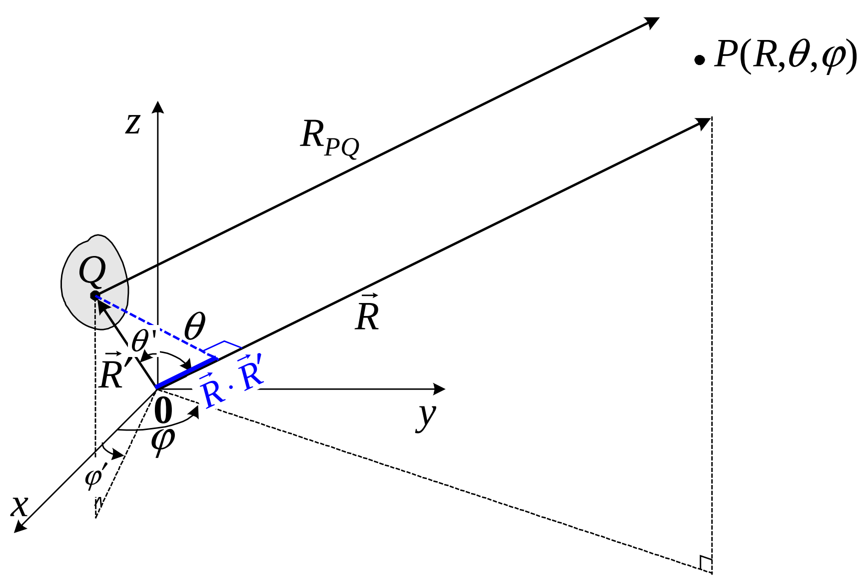


Figure 2: Far-field approximation of the original problem.

We will now apply the far-field approximation to the vector potential in (2.50). Since R depends only on P:

$$\vec{A}(P) = \mu \frac{e^{-jkR}}{4\pi R} \iiint_V \vec{J}(Q) e^{-jk\vec{R}\cdot\vec{R}'} dv_Q \tag{2.56}$$

The integrand (in (2.56)) does not depend on the distance between source and observation point. It depends only on the current distribution in the source volume and the angle between the position vector of the integration point $\vec{R}'$ and the position vector of the observation point $\vec{R}$. This finally explains the general equation for the far-field vector potential in (2.54).

The far-field approximation of the vector potential leads to much simpler equations for the far-field vectors. Assume that there are only electrical currents radiating. Then the EM field is fully described only by the magnetic vector potential $\vec{A}$. One has to substitute (2.54) into the equations of Table 2.1, where $\vec{F}=0$:

$$\begin{aligned} \vec{E} &= -j\omega\vec{A} - \frac{j}{\omega\mu\varepsilon}\nabla\nabla\cdot\vec{A} \\ \vec{H} &= \frac{1}{\mu}\nabla\times\vec{A} \end{aligned} \tag{2.57}$$

The differential operators $\nabla\times$ and $\nabla\nabla\cdot$ have to be expressed in spherical coordinates. All terms decreasing with the distance as $1/R^2$ and faster are neglected. What remains is as follows.

$$\vec{E} = \frac{1}{R}\left\{-j\omega e^{-jkR}\left[\hat{\theta}A_\theta(\theta,\varphi)+\hat{\varphi}A_\varphi(\theta,\varphi)\right]\right\}+\frac{1}{R^2}\{\ \}+\cdots,\quad R\to\infty \tag{2.58}$$

$$\vec{H} = \frac{1}{R}\left\{j\frac{\omega}{\eta}e^{-jkR}\left[\hat{\theta}A_\varphi(\theta,\varphi)-\hat{\varphi}A_\theta(\theta,\varphi)\right]\right\}+\frac{1}{R^2}\{\ \}+\cdots,\quad R\to\infty \tag{2.59}$$

Here, $\eta=\sqrt{\mu/\varepsilon}$ denotes the wave impedance of the medium. One can write equations (2.58) and (2.59) in a more compact way as:

$$\left.\begin{aligned} E_R &\simeq 0 \\ E_\theta &\simeq -j\omega A_\theta \\ E_\varphi &\simeq -j\omega A_\varphi \end{aligned}\right\} \Rightarrow \vec{E}^A \simeq -j\omega\vec{A},\quad E_R^A \simeq 0 \tag{2.60}$$

$$\left.\begin{aligned} H_R &\simeq 0 \\ H_\theta &\simeq +j\frac{\omega}{\eta}A_\varphi = -\frac{E_\varphi}{\eta} \\ H_\varphi &\simeq -j\frac{\omega}{\eta}A_\theta = +\frac{E_\theta}{\eta} \end{aligned}\right\} \Rightarrow \vec{H}^A \simeq -j\frac{\omega}{\eta}\hat{R}\times\vec{A} = \frac{1}{\eta}\hat{R}\times\vec{E}^A \tag{2.61}$$

In an analogous manner, one can obtain the relations between the field vectors and the electric vector potential $\vec{F}$, when magnetic sources only are present.

$$\left.\begin{array}{l} H_R \simeq 0 \\ H_\theta \simeq -j\omega F_\theta \\ H_\varphi \simeq -j\omega F_\varphi \end{array}\right\} \Rightarrow \vec{H}^F \simeq -j\omega \vec{F}, \quad H_R^F \simeq 0 \tag{2.62}$$

$$\left.\begin{array}{l} E_R \simeq 0 \\ E_\theta \simeq -j\omega\eta F_\varphi = \eta H_\varphi \\ E_\varphi \simeq +j\omega\eta F_\theta = -\eta H_\theta \end{array}\right\} \Rightarrow \vec{E}^F \simeq j\omega\eta \hat{R}\times\vec{F} = -\eta\hat{R}\times\vec{H}^F \tag{2.63}$$

The far field of any antenna (any source) has the following important features, which become obvious from equations (2.60) through (2.63):

- The far field has no radial components, $E_R = H_R = 0$. Since the radial direction is also the direction of propagation, the far field is a typical TEM (Transverse Electro-Magnetic) wave.
- The $\vec{E}$ vector and the $\vec{H}$ vector are mutually orthogonal, both of them being also orthogonal to the direction of propagation.
- The magnitudes of the electric field and the magnetic field are related always as $|\vec{E}| = \eta |\vec{H}|$.

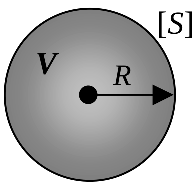
APPENDIX I

Consider the equation

$$\nabla^2\Phi + k^2\Phi = -f\,\delta(x)\delta(y)\delta(z) \qquad \text{(A-1)}$$

Integrate (A-1) within a sphere with its center at (0,0,0) and a radius *R*:

$$\iiint\limits_{V[S]} \nabla^2\Phi dv + \iiint\limits_{V[S]} k^2\Phi dv = -f$$



The field Φ, which is due to this point source, has a spherical symmetry, i.e. it depends on *r* only: $\Phi = C\dfrac{e^{-jkr}}{r}$, where *C* is the constant to be determined. Consider first the integral:

$$I_1 = \iiint\limits_{V[S]} k^2\Phi dv \qquad \text{(A-3)}$$

$$I_1 = \iiint\limits_{V[S]} k^2 C\frac{e^{-jkr}}{r} dv = \int_0^R \int_0^{2\pi} \int_0^{\pi} k^2 C\frac{e^{-jkr}}{r} r^2 \sin\theta d\theta d\varphi dr$$

$$I_1(R) = j4\pi kC\left(R\cdot e^{-jkR} + \frac{e^{-jkR}}{jk} - \frac{1}{jk}\right)$$

To evaluate the integrals in the point of singularity (0,0,0), we let $R \to 0$, i.e. we let the sphere collapse into a point. It is obvious that

$$\lim_{R\to 0} I_1(R) = 0 \qquad \text{(A-4)}$$

Secondly, consider the integral:

$$I_2 = \iiint\limits_{V} \nabla^2\Phi dv = \iiint\limits_{V} \nabla\cdot(\nabla\Phi)\, dv = \oiint\limits_{S} \nabla\Phi d\vec{s} \qquad \text{(A-5)}$$

Here, $d\vec{s} = R^2 \sin\theta dr d\theta d\varphi \hat{r}$ is a surface element on [*S*], and

$$\nabla\Phi = \frac{\partial\Phi}{\partial r}\hat{r} = -C\left(jk\frac{e^{-jkr}}{r} + \frac{e^{-jkr}}{r^2}\right)\hat{r}$$

$$\Rightarrow I_2(R) = -C\left(jkR\cdot e^{-jkR} + e^{-jkR}\right)\int_0^{\pi}\int_0^{2\pi} \sin\theta d\varphi d\theta$$

$$\lim_{R\to 0} I_2(R) = -4\pi C \qquad \text{(A-6)}$$

Substituting (A-4) and (A-6) into (A-2) and taking $\lim_{R\to 0}$, yields:

$$C = \frac{f}{4\pi} \qquad \text{(A-7)}$$

In equation (2.43), the source function is $f = \mu J_z$, that is why $C_1 = (1/4\pi)\mu J_z$.