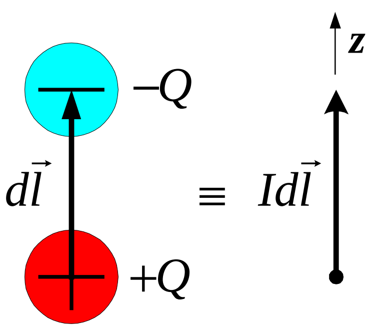
LECTURE 3: Radiation from Infinitesimal (Elementary) Sources

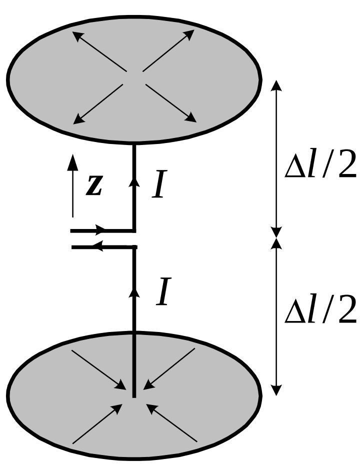
(*Radiation from an infinitesimal dipole. Duality in Maxwell's equations. Radiation from an infinitesimal loop. Radiation zones.*)

1. Radiation from an infinitesimal dipole (current element).

Definition: The infinitesimal dipole is a dipole whose length dl is much smaller than the wavelength λ of the excited wave, i.e. $dl \ll \lambda$ ($dl < \lambda / 50$). The infinitesimal dipole is equivalent to a current element $I d\vec{l}$, where $I d\vec{l} = -\frac{dQ}{dt} d\vec{l}$.



A current element is best illustrated by a very short (compared to λ) piece of infinitesimally thin wire with current *I*. Since the current element is very short, the current is assumed constant along $d\vec{l}$. The ideal current element is practically unrealizable, but a very good approximation of it is the short top-hat antenna. To realize a uniform current distribution along the wire, capacitive plates are used to provide enough charge storage at the end of the wire, so that current is not zero there.



1.1. Magnetic vector potential due to current element radiation.

The magnetic vector potential (VP) $\vec{A}$ due to a linear source is (see eqn. 2.55, Lecture 2):

$$\vec{A}(P) = \int_L \mu_0 I(Q) \frac{e^{-j\beta r}}{4\pi r} d\vec{l}_Q \tag{3.1}$$

Since *I* is constant along Δl,

$$\vec{A} = \mu_0 I \Delta l \frac{e^{-j\beta r}}{4\pi r} \hat{z} \tag{3.2}$$

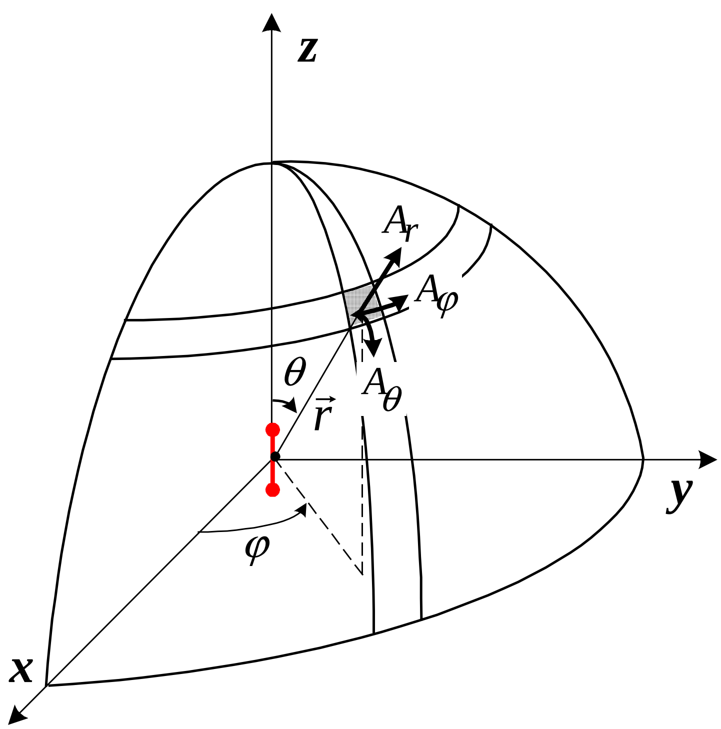
Equation (3.2) gives the field due to an electric current element (infinitesimal dipole) expressed via the magnetic VP $\vec{A}$. ***The field radiated by any complex antenna in linear medium can be represented as a superposition of the fields due to the current elements on the antenna surface***.

The $\vec{A}$ vector will now be represented with its spherical components. In antenna theory, the preferred coordinate system is the spherical one. This is mostly because the ***far field*** radiation is of most significant interest, i.e. the field is analyzed so very far from the source, that it is assumed to propagate only radially away from the source. The transformation from rectangular to spherical coordinates is given by:

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\varphi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \tag{3.3}$$

Applying transformation (3.3) to the $\vec{A}$ vector in (3.2) produces:

$$\begin{aligned} A_r &= A_z\cos\theta = \mu_0 I \vartriangle l \frac{e^{-j\beta r}}{4\pi r}\cos\theta \\ A_\theta &= -A_z\sin\theta = -\mu_0 I \vartriangle l \frac{e^{-j\beta r}}{4\pi r}\sin\theta \\ A_\varphi &= 0 \end{aligned} \tag{3.4}$$



Note that:

1) $\vec{A}$ does not depend on φ (which is due to the cylindrical symmetry of the dipole);

2) the dependence on r, which is $(e^{-j\beta r}/r)$, is separable from the dependence on θ.

1.2. Field vectors due to current element radiation.

Let us now find the field vectors $\vec{H}$ and $\vec{E}$.

a) $\vec{H} = \frac{1}{\mu}\nabla\times\vec{A}$ (3.5)

The curl operator $\nabla\times$ is expressed in spherical coordinates to obtain:

$$\vec{H} = \frac{1}{\mu}\left[\frac{1}{R}\frac{\partial}{\partial r}(r\cdot A_\theta) - \frac{\partial A_r}{\partial \theta}\right]\hat{\varphi} \tag{3.6}$$

The magnetic field $\vec{H}$ has only a φ-component. Finally,

$$\left|\begin{aligned} &H_\varphi = j\beta\cdot(I\vartriangle l)\cdot\sin\theta\cdot\left(1+\frac{1}{j\beta r}\right)\frac{e^{-j\beta r}}{4\pi r} \\ &H_\theta = H_r = 0 \end{aligned}\right. \tag{3.7}$$

b) $\vec{E} = \frac{1}{j\omega\varepsilon}\nabla\times\vec{H} = -j\omega\vec{A} - \frac{j}{\omega\mu\varepsilon}\nabla\nabla\cdot\vec{A}$ (3.8)

Explicitly, in spherical coordinates one finds:

$$\left|\begin{aligned} &E_r = \eta\frac{(I\vartriangle l)\cos\theta}{2\pi r^2}\left(1+\frac{1}{j\beta r}\right)e^{-j\beta r} \\ &E_\theta = j\eta\frac{\beta(I\vartriangle l)\sin\theta}{4\pi r}\left[1+\frac{1}{j\beta r}-\frac{1}{(\beta r)^2}\right]e^{-j\beta r} \\ &E_\varphi = 0 \end{aligned}\right. \tag{3.9}$$

Note: 1) Equations (3.7) and (3.9) show that the EM field generated by the current element is rather complicated unlike the VP $\vec{A}$. The advantage of using the VP instead of the field vectors is obvious even in this simplest example.

2) The field vectors contain terms, which depend on the distance from the source as $(1/r)$, $(1/r^2)$ and $(1/r^3)$, and, therefore, some of them can be neglected at large distances from the dipole.
3) The longitudinal ($\hat{r}$) components of the field vectors decrease fast as the field propagates away from the source (as $1/r^2$ and $1/r^3$ only).
4) The non-zero transverse field components, E_θ and H_φ, are orthogonal to each other, and they have terms, which depend on the distance as $1/r$. These terms differ by a factor of η. They represent the so-called *far field*. The concept of far field will be re-visited later, when the radiation zones are defined.

1.3. Power density and overall radiated power of the infinitesimal dipole.

The complex vector of Poynting $\vec{P}$ describes the complex power density flux. It is calculated as

$$\vec{P} = \frac{1}{2}\left(\vec{E}\times\vec{H}^*\right) = \frac{1}{2}\left(E_r\hat{r} + E_\theta\hat{\theta}\right)\times\left(H_\varphi^*\hat{\varphi}\right) = \frac{1}{2}\left(E_\theta H_\varphi^*\hat{r} - E_r H_\varphi^*\hat{\theta}\right) \qquad (3.10)$$

Substituting (3.7) and (3.9) into (3.10) yields:

$$\left| \begin{aligned} P_r &= \frac{\eta}{8}\left(\frac{I\Delta l}{\lambda}\right)^2 \frac{\sin^2\theta}{r^2}\left[1 - j\frac{1}{(\beta r)^3}\right] \\ P_\theta &= j\eta\beta\frac{(I\Delta l)^2\cos\theta\sin\theta}{16\pi^2 r^3}\left[1 + \frac{1}{(\beta r)^2}\right] \end{aligned} \right. \qquad (3.11)$$

The overall power Π will be calculated over a sphere, and, therefore, only the radial component of the vector of Poynting P_r will have contribution:

$$\Pi = \oint\!\!\oint_S \vec{P}\cdot d\vec{s} = \oint\!\!\oint_S \left(P_r\hat{r} + P_\theta\hat{\theta}\right)\cdot\hat{r}\cdot r^2\sin\theta d\theta d\varphi \qquad (3.12)$$

$$\Pi = \frac{\pi}{3}\eta\left(\frac{I\Delta l}{\lambda}\right)^2\left[1 - \frac{j}{(\beta r)^3}\right] \qquad (3.13)$$

The radiated power is equal to the real part of the complex power (the time-average of the total power flow, see Lecture 2, Section 2). Therefore, the radiated power of an infinitesimal electric dipole is:

$$\Pi_{rad} = \frac{\pi}{3}\eta\left(\frac{I\Delta l}{\lambda}\right)^2 \qquad (3.14)$$

Here, it is appropriate to introduce the concept of radiation resistance R_r, which can describe the power loss due to radiation in an equivalent circuit of the antenna:

$$\Pi = \frac{1}{2} R_r I^2 \Rightarrow R_r = \frac{2\Pi}{I^2}, \Rightarrow R_r^{id} = \frac{2\pi}{3}\eta\left(\frac{\Delta l}{\lambda}\right)^2$$

The radiation resistance will be discussed later in much more detail.

2. Duality in Maxwell's equations.

Duality in Electromagnetics means that the EM field is described by two sets of quantities, which correspond to each other in such a manner that substituting the quantities from one set with the respective quantities from the other set in any given equation produces a valid equation (the dual of the given one).

We shall deduce these dual sets by simple comparison of Maxwell's equations describing two dual fields: the field of electric sources, and the field of magnetic sources. As a word of caution, duality exists even if there are no sources present in the region of interest. Table 2.1 is just an easy intuitive way to illustrate duality and to define the dual sets of EM quantities.

TABLE 2.1. DUALITY IN ELECTROMAGNETIC EQUATIONS

Electric sources $(\vec{J} \neq 0, \vec{M} = 0)$	Magnetic sources $(\vec{J} = 0, \vec{M} \neq 0)$
$\nabla \times \vec{E} = -j\omega\mu\vec{H}$	$\nabla \times \vec{H} = j\omega\varepsilon\vec{E}$
$\nabla \times \vec{H} = j\omega\varepsilon\vec{E} + \vec{J}$	$\nabla \times \vec{E} = -j\omega\mu\vec{H} - \vec{M}$
$\nabla \cdot \vec{D} = \rho$	$\nabla \cdot \vec{B} = \rho_m$
$\nabla \cdot \vec{B} = 0$	$\nabla \cdot \vec{D} = 0$
$\nabla \cdot \vec{J} = -j\omega\rho$	$\nabla \cdot \vec{M} = -j\omega\rho_m$
$\nabla^2\vec{A} + \beta^2\vec{A} = -\mu\vec{J}$	$\nabla^2\vec{F} + \beta^2\vec{F} = -\varepsilon\vec{M}$
$\vec{A} = \iiint_V \mu\vec{J}\frac{e^{-j\beta R}}{4\pi R}dv$	$\vec{F} = \iiint_V \varepsilon\vec{M}\frac{e^{-j\beta R}}{4\pi R}dv$
$\vec{H} = \frac{1}{\mu}\nabla\times\vec{A}$	$\vec{E} = -\frac{1}{\varepsilon}\nabla\times\vec{F}$
$\vec{E} = -j\omega\vec{A} - \frac{j}{\omega\mu\varepsilon}\nabla\nabla\cdot\vec{A}$	$\vec{H} = -j\omega\vec{F} - \frac{j}{\omega\mu\varepsilon}\nabla\nabla\cdot\vec{F}$

TABLE 2.2. DUAL QUANTITIES IN ELECTROMAGNETICS

given	dual
$\vec{E}$	$\vec{H}$
$\vec{H}$	$-\vec{E}$
$\vec{J}$	$\vec{M}$
$\vec{M}$	$-\vec{J}$
$\vec{A}$	$\vec{F}$
$\vec{F}$	$-\vec{A}$
ε	μ
μ	ε

Additional:

$\eta \to 1/\eta$

$1/\eta \to \eta$

$\beta \to \beta$

3. Radiation from an infinitesimal magnetic dipole (electric current loop).

3.1. The vector potential and the field vectors of a magnetic dipole (magnetic current element) $I_m \triangle l$.

Using the duality theorem, the field of a magnetic dipole is readily found by simple substitution of the dual quantities in equations (3.4), (3.7) and (3.9) according to Table 2.2. We shall denote the magnetic current, which is the dual of the electric current *I*, by I_m (measured in *Volts*).

$$\begin{aligned} F_r &= F_z \cos\theta = \varepsilon_0 \left(I_m \triangle l\right) \frac{e^{-j\beta r}}{4\pi r} \cos\theta \\ F_\theta &= -F_z \sin\theta = -\varepsilon_0 \left(I_m \triangle l\right) \frac{e^{-j\beta r}}{4\pi r} \sin\theta \\ F_\varphi &= 0 \end{aligned} \tag{3.15}$$

$$\begin{aligned} E_\varphi &= -\frac{j\beta \cdot (I_m \triangle l) \cdot \sin\theta}{4\pi r} \left(1 + \frac{1}{j\beta r}\right) e^{-j\beta r} \\ E_\theta &= E_r = 0 \end{aligned} \tag{3.16}$$

$$\begin{aligned} H_r &= \frac{1}{\eta} \frac{\left(I_m \triangle l\right)\cos\theta}{2\pi r^2} \left(1 + \frac{1}{j\beta r}\right) e^{-j\beta r} \\ H_\theta &= \frac{j}{\eta} \frac{\beta \left(I_m \triangle l\right)\sin\theta}{4\pi r} \left(1 + \frac{1}{j\beta r} - \frac{1}{\beta^2 r^2}\right) e^{-j\beta r} \\ H_\varphi &= 0 \end{aligned} \tag{3.17}$$

3.2. Equivalence between a magnetic dipole (magnetic current element) and an electric current loop.

First, we shall prove the equivalence of the fields excited by the following sources

$$j\omega\mu\vec{J} \Leftrightarrow \nabla\times\vec{M} \tag{3.18}$$

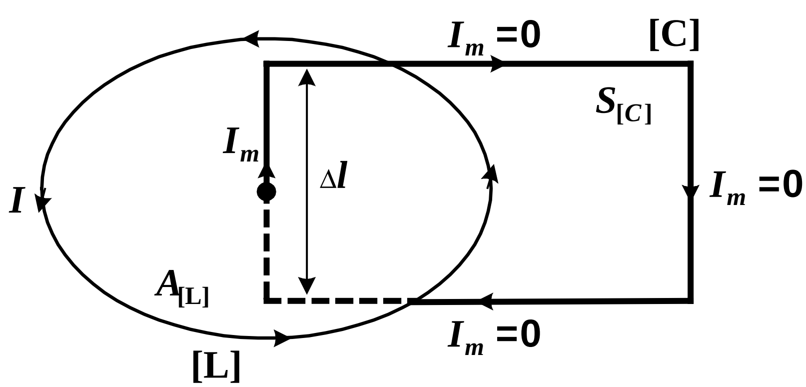
$$\left| \begin{array}{l} -\nabla\times\vec{E}_1 = j\omega\mu\vec{H}_1 \\ \nabla\times\vec{H}_1 = j\omega\varepsilon\vec{E}_1 + \vec{J} \Rightarrow \end{array} \right. \quad (3.19) \qquad \nabla\times\nabla\times\vec{E}_1 - \omega^2\mu\varepsilon\vec{E}_1 = -j\omega\mu\vec{J} \quad (3.20)$$

$$\left| \begin{array}{l} -\nabla\times\vec{E}_2 = j\omega\mu\vec{H}_2 + \vec{M} \\ \nabla\times\vec{H}_2 = j\omega\varepsilon\vec{E}_2 \qquad \Rightarrow \end{array} \right. \quad (3.21) \qquad \nabla\times\nabla\times\vec{E}_2 - \omega^2\mu\varepsilon\vec{E}_2 = -\nabla\times\vec{M} \quad (3.22)$$

If the boundary conditions (BCs) for $\vec{E}_1$ of problem (3.20) are the same as the BCs for $\vec{E}_2$ in problem (3.22), and the excitations of both fields fulfill

$$j\omega\mu\vec{J} = \nabla\times\vec{M}\ , \tag{3.23}$$

then both fields will be identical, i.e. $\vec{E}_1 \equiv \vec{E}_2$ and $\vec{H}_1 \equiv \vec{H}_2$.



Consider a loop [L] of electric current *I*. Equation (3.23) can be written in integral form as:

$$j\omega\mu \iint_{S_{[C]}} \vec{J}d\vec{s} = \oint_C \vec{M}d\vec{c} \tag{3.24}$$

The integral at the left-hand side is the electric current *I*. $\vec{M}$ is assumed non-zero and constant only at the section (Δl), which is normal to the loop's plane and passes through the loop's center. Then,

$$j\omega\mu I = M\Delta l \tag{3.25}$$

The magnetic current I_m corresponding to the loop [L] will be obtained by multiplying the magnetic current density $\vec{M}$ by the area of the loop $A_{[L]}$, which yields

$$j\omega\mu IA_{[L]} = I_m \Delta l \tag{3.26}$$

It was shown that a small loop of electric current I and of area $A_{[L]}$ creates EM field equivalent to that of a small magnetic dipole (magnetic current element) $(I_m \Delta l)$, *such that* (3.26) *holds*. Here, it was assumed that the electric current is constant along the loop, which is true only for very small loops ($a < 0.01\lambda$, where a is the loop's radius). If the loop is larger, then the field expressions provided below will be inaccurate, and other solutions should be used.

3.3. VP and field vectors of an infinitesimal loop antenna.

The expressions below are derived simply by inserting (3.26) into (3.16)-(3.17).

$$E_\varphi = -\eta\beta^2 \frac{(IA)\cdot\sin\theta}{4\pi r}\left(1+\frac{1}{j\beta r}\right)e^{-j\beta r} \tag{3.27}$$

$$H_r = j\beta \frac{(IA)\cdot\cos\theta}{2\pi r^2}\left(1+\frac{1}{j\beta r}\right)e^{-j\beta r} \tag{3.28}$$

$$H_\theta = -\beta^2 \frac{(IA)\cdot\sin\theta}{4\pi r}\left(1+\frac{1}{j\beta r}-\frac{1}{\beta^2 r^2}\right)e^{-j\beta r} \tag{3.29}$$

$$E_r = E_\theta = H_\varphi = 0 \tag{3.30}$$

The far-field terms (which have $1/r$ dependence on the distance from the source) show the same behaviour as in the case of an infinitesimal dipole antenna: the electric field E_φ is orthogonal to the magnetic field H_θ and differs just by a factor of η; the longitudinal $\hat{r}$ components have no far-field terms. The dependence of the Poynting vector and the complex power on the distance r is the same as in the case of an infinitesimal electric dipole. The radiated power can be found to be:

$$\Pi_{rad} = \frac{1}{12\pi}\eta\beta^4 (IA)^2 \tag{3.31}$$

4. Radiation zones – basic concepts.

The space surrounding the antenna is divided into three regions according to the predominant field behaviour. The boundaries between the regions are not distinct and the field behaviour changes very gradually as these boundaries are crossed. In this course, we shall be mostly concerned with the far-field characteristics of the antennas.

4.1. Reactive near-field region

This is the region immediately surrounding the antenna, where the reactive field predominates. For most antennas, it is assumed that this region is a sphere with the antenna at its centre, and with a radius of

$$r \approx 0.62\sqrt{\frac{D^3}{\lambda}}, \tag{3.32}$$

where D is the largest dimension of the antenna, and λ is the wavelength of the radiated field. The above expression will be derived in Section 5. It must be noted that this limit is most appropriate for wire and waveguide aperture antennas, while it is not valid for electrically large reflector antennas.

At this point, we shall discuss the general field behaviour making use of our knowledge of the infinitesimal dipole field. When (3.32) is true, it is also true for most antennas that $\beta r \ll 1$. Then, the most significant terms in the field expressions (3.7) and (3.9) will be

$$\left|\begin{array}{l} H_\varphi \approx \dfrac{(I\Delta l)e^{-j\beta r}}{4\pi r^2}\sin\theta \\ E_\theta \approx -j\eta\dfrac{(I\Delta l)e^{-j\beta r}}{4\pi\beta r^3}\sin\theta \\ E_r \approx -j\eta\dfrac{(I\Delta l)e^{-j\beta r}}{4\pi\beta r^3}\cos\theta \\ H_r = H_\theta = E_\varphi = 0 \end{array}\right. , \quad \beta r \ll 1 \tag{3.33}$$

This approximated field is purely reactive ($\vec{H}$ and $\vec{E}$ are in phase quadrature). Actually, the $e^{-j\beta r}$ can be neglected, and after some simple mathematical transformations it can be clearly shown that: 1) the H_φ component is exactly the magnetostatic field of a current filament $(I\Delta l)$; 2) the E_θ and E_r components are exactly the electrostatic field of a dipole.

That the field is purely reactive at points, which are very close to the infinitesimal dipole, is obvious from the equation (3.13) describing the total complex power. Its imaginary part is:

$$\mathrm{Im}\{\Pi\} = -\frac{\pi}{3}\eta\left(\frac{I\Delta l}{\lambda}\right)^2 \frac{1}{(\beta r)^3} \tag{3.34}$$

$\mathrm{Im}\{\Pi\}$ will obviously dominate over the radiated power

$$\mathrm{Re}\{\Pi\} = \frac{\pi}{3}\eta\left(\frac{I\Delta l}{\lambda}\right)^2 = \Pi_{rad}, \tag{3.35}$$

when $r \to 0$, since Π_{rad} does not depend on *r* at all. In the near-field region, the radial reactive power flow density P_r has predominant magnetic character (negative imaginary number) and decreases as $(1/r^5)$:

$$P_r^{near} = -j\frac{\eta}{8}\left(\frac{I\Delta l}{\lambda}\right)^2 \frac{\sin^2\theta}{\beta^3 r^5} \tag{3.36}$$

The P_θ has the same order of dependence on *r* but has predominant electric character:

$$P_\theta^{near} = j\eta\beta \frac{(I\Delta l)^2 \cos\theta \sin\theta}{16\pi^2 r^3}\left[1 + \frac{1}{(\beta r)^2}\right] \tag{3.37}$$

Reminder: According to Poynting's theorem

$$-\frac{1}{2}\left(\vec{H}^* \cdot \vec{M}^i + \vec{E}\cdot\vec{J}^{*i}\right) = \frac{1}{2}\nabla\cdot\left(\vec{E}\times\vec{H}^*\right) + \frac{1}{2}\sigma\left|\vec{E}\right|^2 + j\omega 2\left(\frac{1}{4}\mu\left|\vec{H}\right|^2 - \frac{1}{4}\varepsilon\left|\vec{E}\right|^2\right)$$

or

$$p_s = p_{rad} + p_{loss} + 2j\omega\left(\overline{w}_m - \overline{w}_e\right)$$

where:

$p_s = -\frac{1}{2}\left(\vec{H}^* \cdot \vec{M}^i + \vec{E}\cdot\vec{J}^i\right)$ is the supplied complex power density, W/m^3;

$p_{rad} = \frac{1}{2}\nabla\cdot\left(\vec{E}\times\vec{H}^*\right) = \nabla\cdot\vec{P}$ is the complex power density entering or leaving the point, W/m^3;

$p_{loss} = \frac{1}{2}\sigma\left|\vec{E}\right|^2$ is the loss power density (real only), W/m^3;

$\overline{w}_m = \frac{1}{4}\mu\left|\vec{H}\right|^2$ is the time-average magnetic energy density, J/m^3;

$\bar{w}_e = \frac{1}{4}\varepsilon|\vec{E}|^2$ is the time-average electric energy density, J/m^3.

4.2. Radiating near-field (Fresnel) region

This is an intermediate region between the reactive near-field region and the far-field region, where the radiation field predominates but the angular field distribution is still dependent on the distance from the antenna. In this region, $\beta r \geq 1$. For most antennas, it is assumed that the Fresnel region is enclosed between two spherical surfaces:

$$0.62\sqrt{\frac{D^3}{\lambda}} \leq r \leq \frac{2D^2}{\lambda} \tag{3.38}$$

Here, D is the largest dimension of the antenna. This region is called *Fresnel region* because its field expressions reduce to Fresnel integrals.

The fields of an infinitesimal dipole in the Fresnel region are obtained by neglecting the higher-order $(1/r)^n$-terms:

$$\left| \begin{aligned} & H_\varphi \approx \frac{j\beta \cdot (I\Delta l) \cdot e^{-j\beta r}}{4\pi r} \cdot \sin\theta \\ & E_r \approx \eta \frac{(I\Delta l) \cdot e^{-j\beta r}}{2\pi r^2} \cdot \cos\theta \\ & E_\theta \approx j\eta \frac{\beta \cdot (I\Delta l) \cdot e^{-j\beta r}}{4\pi r} \cdot \sin\theta \\ & H_\theta = H_r = E_\varphi = 0 \end{aligned} \right. , \quad \beta r \geq 1 \tag{3.39}$$

The radial component E_r is still not negligible, but the transverse components (E_θ and H_φ) are dominant.

4.3. Far-field (Fraunhofer) region

Only terms $\sim 1/r$ are considered, when $\beta r \gg 1$. The angular field distribution does not depend on the distance from the source any more, i.e. the *far-field pattern* is already well established. The field is a transverse EM wave. For most antennas, the far-field region is defined as:

$$r \geq \frac{2D^2}{\lambda} \tag{3.40}$$

The far-field of the infinitesimal dipole is obtained as:

$$\left| \begin{array}{l} H_\varphi \approx \dfrac{j\beta \cdot (I\Delta l)\cdot e^{-j\beta r}}{4\pi r}\cdot \sin\theta \\ E_\theta \approx j\eta \dfrac{\beta \cdot (I\Delta l)\cdot e^{-j\beta r}}{4\pi r}\cdot \sin\theta,\ \beta r \gg 1 \\ E_r \approx 0 \\ H_\theta = H_r = E_\varphi = 0 \end{array} \right. \tag{3.41}$$

Important features of the far field:

1) no radial components;
2) the angular field distribution is independent of *r*;
3) $\vec{E} \perp \vec{H}$;
4) $\eta = Z_i = \dfrac{E_\theta}{H_\varphi}$;
5) $\vec{P} = \dfrac{1}{2}\left(\vec{E}\times\vec{H}^*\right) = \dfrac{1}{2}\dfrac{|E_\theta|^2}{\eta}\hat{r} = \dfrac{1}{2}\eta |H_\varphi|^2 \hat{r}$. (3.42)

5. Region separation and accuracy of radiation integral approximations.

In most practical cases, the closed form solution of the radiation integral (the VP integral) is impossible. For the evaluation of the far fields or the fields in the Fresnel region, standard approximations are applied, from which the boundaries of these regions are derived.

Consider the VP integral for a linear current source:

$$\vec{A} = \frac{\mu}{4\pi}\int_L \frac{\vec{I}\left(l'\right)}{R} e^{-j\beta R} dl', \tag{3.43}$$

where $R = \sqrt{\left(x-x'\right)^2 + \left(y-y'\right)^2 + \left(z-z'\right)^2}$. The observation point is at $P(x,y,z)$, and the source point is located at $Q(x',y',z')$, which is along the integration contour *L*.

So far, we have analyzed just the infinitesimal dipole, whose current is constant along *L*. In practical antennas, this is rarely true, and the solution of (3.43) can be very complicated depending on the function $\vec{I}\left(l'\right)$. Besides, because of the infinitesimal size of the source, the distance between the integration point and the observation point *R* was considered constant and equal to the distance from the centre of the dipole $R = r = \sqrt{x^2 + y^2 + z^2}$. The closer the observation point to a finite-size antenna, the less accurate this assumption is.

The integral kernel $\frac{e^{-j\beta R}}{R}$ will be divided into two factors: the amplitude factor $(1/R)$, and the phase factor $e^{-j\beta R}$. The amplitude factor is not very sensitive to errors in *R*. In both, the Fresnel and the Fraunhofer regions, the approximation

$$\frac{1}{R} \approx \frac{1}{r} \tag{3.44}$$

is acceptable.

The above approximation is unacceptable in the phase term. ***To keep the phase term error low enough, the maximum error in*** (βR) ***must be kept below*** $\pi/8 = 22.5^\circ$.

Neglect the antenna dimensions along the *x* and the *y*-axes (infinitesimally thin wire). Then,

$$x' = y' = 0 \Rightarrow R = \sqrt{x^2 + y^2 + (z - z')^2} \tag{3.45}$$

$$\Rightarrow R = \sqrt{x^2 + y^2 + z^2 + \left(z'^2 - 2zz'\right)} = \sqrt{r^2 + \left(z'^2 - 2rz' \cdot \cos\theta\right)} \tag{3.46}$$

Using the binomial expansion♣:

$$R \approx \left(r^2\right)^{1/2} + \frac{1}{2}\left(r^2\right)^{-1/2}\left(z'^2 - 2rz'\cos\theta\right) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\left(r^2\right)^{-3/2}\left(z'^2 - 2rz'\cos\theta\right)^2 + \cdots$$

$$R \simeq r - z'\cos\theta + \frac{z'^2}{2r} - \frac{z'^2\cos^2\theta}{2r} + O^2 \tag{3.47}$$

O^2 denotes terms of the order $(1/r^2)$ and higher. Simplifying further, one arrives at:

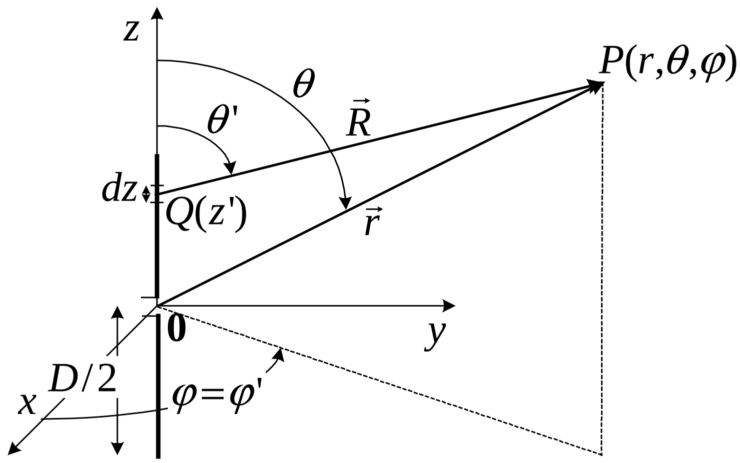
$$R = r - z'\cos\theta + \frac{1}{2r}z'^2\sin^2\theta + O^2 \tag{3.48}$$

♣ $(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots$

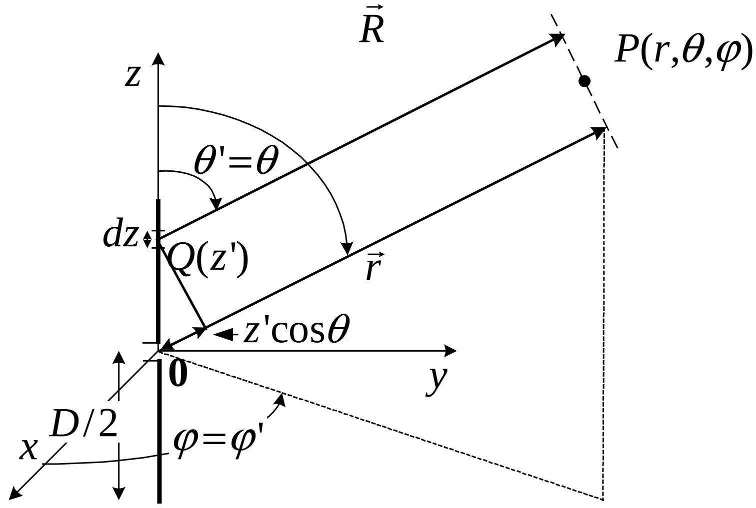
(a) far-field approximation

Only the first two terms in the expansion (3.48) are taken into account:

$$R \approx r - z'\cos\theta \tag{3.49}$$



(a) Finite-size dipole



(b) Finite-size dipole - far-field approximations

The most significant error term in R is

$$e = \frac{1}{2}\frac{z'^2}{r}\sin^2\theta,$$

which has its maximum at $\theta = \pi/2$, $e_{\max} = \frac{z'^2}{2r}$. The consequence is a maximum error in (βR), which has to be kept below $\pi/8$:

$$\beta \cdot \frac{z'^2_{max}}{2r} \le \frac{\pi}{8}$$

z'_{max} is actually half the largest dimension of the antenna, $z'_{max} = D/2$. Now, it is easy to find the smallest distance from the antenna centre *r*, at which the phase error is acceptable:

$$r \ge 2\frac{D^2}{\lambda} \qquad (3.50)$$

This result is identical with the far-zone limit defined in (3.40).

(b) <u>radiating near-field (Fresnel region) approximation</u>

This region is adjacent to the Fraunhofer region, so its upper boundary is specified by:

$$r \le \frac{2D^2}{\lambda} \qquad (3.51)$$

When the observation point belongs to this region, one should take one more term in the expansion of *R* as given by (3.48) to reduce sufficiently the phase error. The approximation this time is:

$$R \approx r - z'\cos\theta + \frac{1}{2r} z'^2 \sin^2\theta \qquad (3.52)$$

The most significant error term is:

$$e = \frac{1}{2}\frac{z'^3}{r^2}\cos\theta \sin^2\theta \qquad (3.53)$$

The angles θ_o must be found, at which *e* has its extrema.

$$\frac{\partial e}{\partial \theta} = \frac{z'^3}{2r^2}\sin\theta\left(-\sin^2\theta + 2\cos^2\theta\right) = 0 \qquad (3.54)$$

The roots of (3.54) are:

$$\theta_o^{(1)} = 0 \to \min$$
$$\theta_o^{(2),(3)} = \arctan\left(\pm\sqrt{2}\right) \approx \pm 54.7^\circ \to \max \qquad (3.55)$$

Following a procedure similar to case (a), one obtains:

$$\beta e_{max} = \frac{2\pi}{\lambda} \cdot \frac{1}{2}\frac{z'^3}{r^2}\cos\theta_o^{(2)} \sin^2\theta_o^{(2)}$$

$$\Rightarrow \beta e_{max} = \frac{\pi}{12\sqrt{3}}\frac{D^3}{\lambda r^2} \le \frac{\pi}{8}$$

$$\Rightarrow r \geq \sqrt{\frac{2}{3\sqrt{3}}\frac{D^3}{\lambda}} = 0.62\sqrt{\frac{D^3}{\lambda}} \tag{3.56}$$

Equation (3.56) states the lower boundary of the Fresnel region (for wire antennas) and is identical to the left-hand side of (3.38).