Lecture 5: Polarization and Related Antenna Parameters

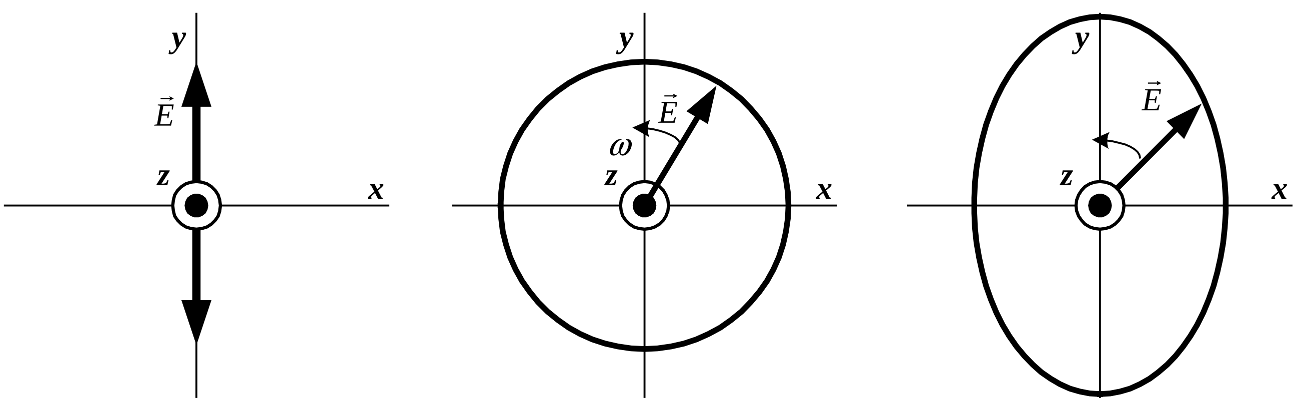
(*Polarization of EM fields – revision. Polarization vector. Antenna polarization. Polarization loss factor and polarization efficiency.*)

1. Polarization of EM fields.

The polarization of the EM field describes the time variations of the field vectors at a given point. In other words, it describes the way the direction and magnitude the field vectors (usually $\vec{E}$) change in time.

The ***polarization*** is the figure traced by the extremity of the time-varying field vector at a given point.

According to the shape of the trace, three types of polarization exist for harmonic fields: ***linear, circular*** and ***elliptical***:



(a) linear polarization (b) circular polarization (c) elliptical polarization

Any type of polarization can be represented by two orthogonal linear polarizations, ($\tilde{E}_x, \tilde{E}_y$) or ($\tilde{E}_H, \tilde{E}_V$), whose fields are out of phase by an angle of δ_L.

- If $\delta_L = 0$ or $n\pi$, then a linear polarization results.
- If $\delta_L = \pi/2$ (90°) and $E_x = E_y$, then a circular polarization results.
- In the most general case, elliptical polarization is defined.

It is also true that any type of polarization can be represented by a right-hand circular and a left-hand circular polarizations ($\tilde{E}_L, \tilde{E}_R$).

We shall revise the above statements and definitions, while introducing the new concept of polarization vector.

2. Field polarization in terms of two orthogonal linearly polarized components.

The polarization of any field can be represented by a suitable set of two orthogonal linearly polarized fields. Assume that locally a far field propagates along the *z*-axis, and the field vectors have only transverse components. Then, the set of two orthogonal linearly polarized fields along the *x*-axis and along the *y*-axis, is sufficient to represent any TEM_z field. We shall use this arrangement to demonstrate the idea of polarization vector.

The field (time-dependent vector or phasor vector) is decomposed into two orthogonal components:

$$\vec{e} = \vec{e}_x + \vec{e}_y \quad \Rightarrow \quad \vec{E} = \vec{E}_x + \vec{E}_y, \tag{5.1}$$

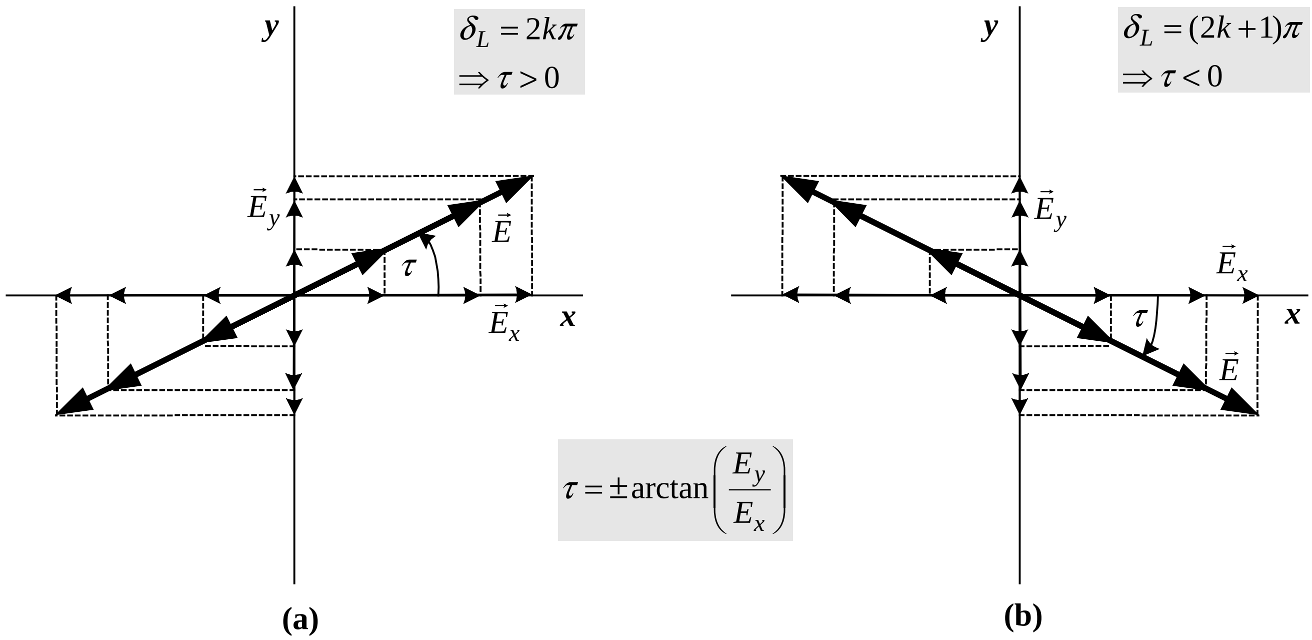
$$\begin{matrix} \vec{e}_x = E_x \cos(\omega t - \beta z)\hat{x} \\ \vec{e}_y = E_y \cos(\omega t - \beta z + \delta_L)\hat{y} \end{matrix} \quad \Rightarrow \quad \begin{matrix} \vec{E}_x = E_x \hat{x} \\ \vec{E}_y = E_y e^{j\delta_L}\hat{y} \end{matrix} \tag{5.2}$$

At a fixed position (assume $z = 0$), equation (5.1) can be written as:

$$\begin{aligned} &\vec{e}(t) = \hat{x}\cdot E_x \cos\omega t + \hat{y}\cdot E_y \cos(\omega t + \delta_L) \\ &\Rightarrow \boxed{\vec{E} = \hat{x}\cdot E_x + \hat{y}\cdot E_y e^{j\delta_L}} \end{aligned} \tag{5.3}$$

<u>Case 1</u>: Linear polarization: $\delta_L = n\pi, \quad n = 0,1,2,\ldots$

$$\begin{aligned} &\vec{e}(t) = \hat{x}\cdot E_x \cos(\omega t) + \hat{y}\cdot E_y \cos(\omega t \pm n\pi) \\ &\Rightarrow \boxed{\vec{E} = \hat{x}\cdot E_x \pm \hat{y}\cdot E_y} \end{aligned} \tag{5.4}$$



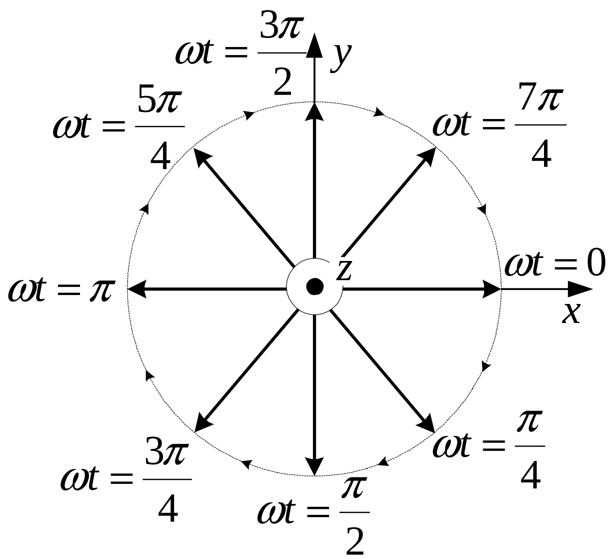
<u>Case 2</u>: Circular polarization:

$$E_x = E_y = E_m \quad \text{and} \quad \delta_L = \pm\left(\frac{\pi}{2} + n\pi\right), \quad n = 0,1,2,\ldots$$

$$\vec{e}(t) = \hat{x} \cdot E_x \cos(\omega t) + \hat{y} \cdot E_y \cos[\omega t \pm (\pi/2 + n\pi)]$$
$$\Rightarrow \boxed{\vec{E} = E_m(\hat{x} \pm \hat{y} \cdot j)} \tag{5.5}$$

$$\vec{E} = E_m(\hat{x} + j\hat{y})$$

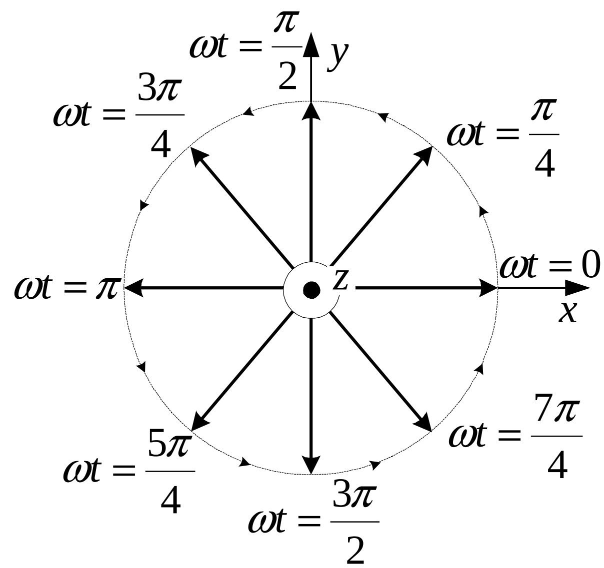
$$\delta_L = +\frac{\pi}{2} + 2n\pi$$



If $(-\hat{z})$ is the direction of propagation: ***clockwise (CW)*** or ***right-hand*** polarization

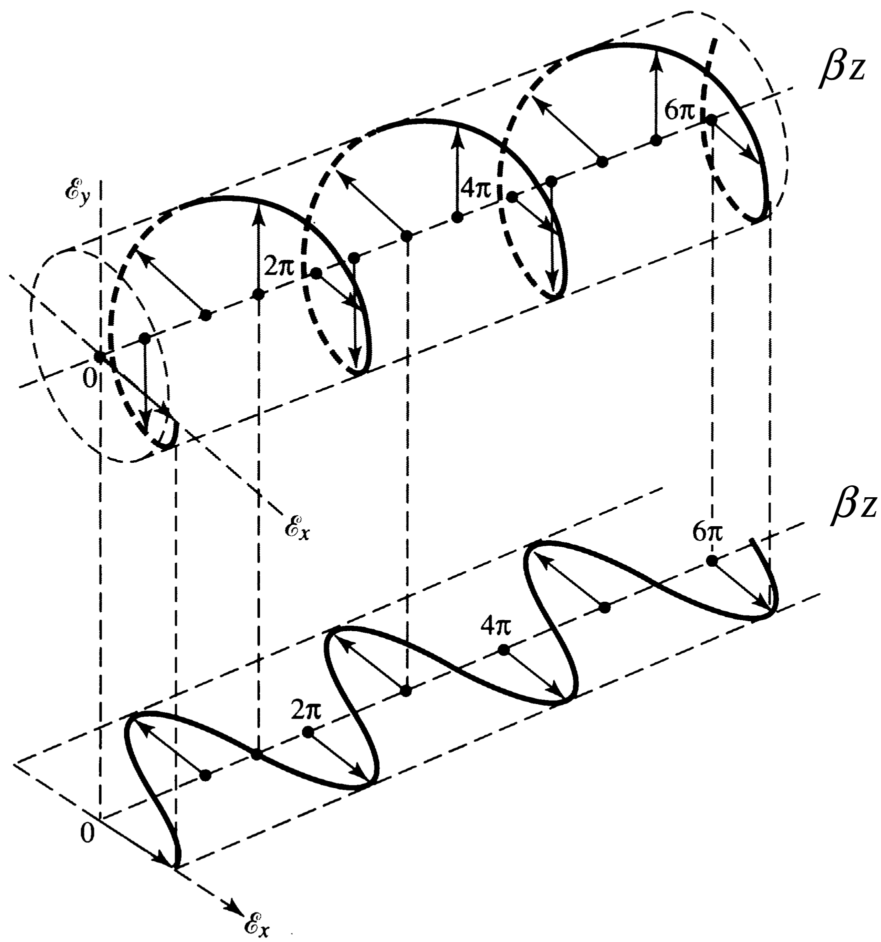
$$\vec{E} = E_m(\hat{x} - j\hat{y})$$

$$\delta_L = -\frac{\pi}{2} - 2n\pi$$



If $(-\hat{z})$ is the direction of propagation: ***counterclockwise (CCW)*** or ***left-hand*** polarization

A picture of the field vector (at a particular moment of time) along the direction of propagation (left-hand circularly polarized wave):



Case 3: Elliptical polarization

The field vector at a given point traces an ellipse as a function of time. This is the most general type of polarization of time-harmonic fields, obtained for any phase difference δ and any ratio (E_x / E_y). Mathematically, the linear and the circular polarizations are special cases of the elliptical polarization. In practice, however, the term *elliptical polarization* is used to indicate polarizations *other than linear or circular*.

$$\begin{aligned} &\vec{e}(t) = \hat{x} \cdot E_x \cos \omega t + \hat{y} \cdot E_y \cos(\omega t + \delta_L) \\ &\Rightarrow \vec{E} = \hat{x} \cdot E_x + \hat{y} \cdot E_y e^{j\delta_L} \end{aligned} \tag{5.6}$$

Show that the trace of the time-dependent vector is an ellipse:

$$e_y(t) = E_y(\cos \omega t \cdot \cos \delta_L - \sin \omega t \cdot \sin \delta_L)$$

$$\cos \omega t = \frac{e_x(t)}{E_x} \text{ and } \sin \omega t = \sqrt{1 - \left(\frac{e_x(t)}{E_x} \right)^2}$$

$$\sin^2 \delta_L = \left[\frac{e_x(t)}{E_x} \right]^2 - 2 \left[\frac{e_x(t)}{E_x} \right] \left[\frac{e_y(t)}{E_y} \right] \cos \delta_L + \left[\frac{e_y(t)}{E_y} \right]^2$$

or

$$1 = x^2(t) - 2x(t)y(t)\cos \delta_L + y^2(t), \tag{5.7}$$

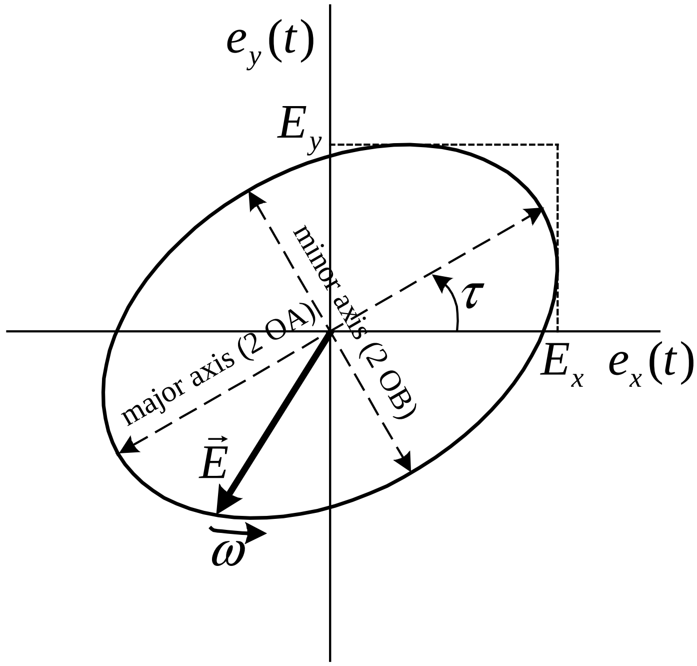
where:

$$x(t) = \frac{e_x(t)}{E_x \sin \delta_L} = \frac{\cos \omega t}{\sin \delta_L};$$

$$y(t) = \frac{e_y(t)}{E_y \sin \delta_L} = \frac{\cos(\omega t + \delta_L)}{\sin \delta_L}$$

Equation (5.7) is the parametric equation of an ellipse centered in the $x-y$ plane. It describes the motion of a point of coordinates $e_x(t)$ and $e_y(t)$ along an ellipse with a frequency ω.

The elliptical polarization can also be *right-hand* and *left-hand* polarization, depending on the relation between the direction of propagation and the direction of rotation.



The parameters of the polarization ellipse are given below. Their derivation is given in Appendix I.

a) major axis ($2\times$OA)

$$\text{OA} = \sqrt{\frac{1}{2}\left[E_x^2 + E_y^2 + \sqrt{E_x^4 + E_y^4 + 2E_x^2E_y^2\cos(2\delta_L)}\right]} \tag{5.8}$$

b) minor axis ($2\times$OB)

$$\text{OB} = \sqrt{\frac{1}{2}\left[E_x^2 + E_y^2 - \sqrt{E_x^4 + E_y^4 + 2E_x^2E_y^2\cos(2\delta_L)}\right]} \tag{5.9}$$

c) tilt angle τ

$$\tau = \frac{1}{2}\arctan\left(\frac{2E_xE_y}{E_x^2 - E_y^2}\cos\delta_L\right) \tag{5.10}$$

d) axial ratio

$$AR = \frac{\text{major axis}}{\text{minor axis}} = \frac{\text{OA}}{\text{OB}} \tag{5.11}$$

Note: The linear and circular polarizations can be considered as special cases of the elliptical polarization.

- If $\delta_L = \pm\left(\frac{\pi}{2} + 2n\pi\right)$ and $E_x = E_y$, then $\text{OA} = \text{OB} = E_x = E_y$; the ellipse becomes a circle.

- If $\delta_L = n\pi$, then $\text{OB} = 0$ and $\tau = \pm \arctan\left(\frac{E_x}{E_y}\right)$; the ellipse collapses into a line.

3. Field polarization in terms of two circularly polarized components

The representation of a complex vector field in terms of circularly polarized components is somewhat less easy to perceive but it is actually more useful in the calculation of the polarization ellipse parameters.

$$\vec{E} = \tilde{E}_R(\hat{x} + j\hat{y}) + \tilde{E}_L(\hat{x} - j\hat{y}) \qquad (5.12)$$

Assuming a relative phase difference of $\delta_C = \varphi_L - \varphi_R$, one can write (5.12) as:

$$\vec{E} = E_R(\hat{x} + j\hat{y}) + E_L e^{j\delta_C}(\hat{x} - j\hat{y}) \qquad (5.13)$$

The relation between the linear-component and the circular-component representations of the field polarization is easily found as:

$$\vec{E} = \hat{x}\underbrace{(\tilde{E}_R + \tilde{E}_L)}_{\tilde{E}_x} + \hat{y}\underbrace{j(\tilde{E}_R - \tilde{E}_L)}_{\tilde{E}_y} \qquad (5.14)$$

4. Polarization vector and polarization ratio

The *polarization vector* is the normalized phasor of the electric field vector. It is a complex-number vector of unit magnitude and direction coinciding with the direction of the electric field vector.

$$\hat{\rho}_L = \frac{\vec{E}}{E_m} = \hat{x} \cdot \frac{E_x}{E_m} + \hat{y} \cdot \frac{E_y}{E_m} e^{j\delta_L}, \quad E_m = \sqrt{E_x^2 + E_y^2} \qquad (5.15)$$

The polarization vector takes the following forms in some special cases:

Case 1: Linear polarization

$$\hat{\rho}_L = \hat{x} \cdot \frac{E_x}{E_m} \pm \hat{y} \cdot \frac{E_y}{E_m}, \quad E_m = \sqrt{E_x^2 + E_y^2} \qquad (5.16)$$

Case 2: Circular polarization

$$\hat{\rho}_L = \frac{1}{\sqrt{2}}(\hat{x} \pm \hat{y} \cdot j), \quad E_m = \sqrt{2} \cdot E_x = \sqrt{2} \cdot E_y \tag{5.17}$$

The *polarization ratio* is the ratio of the phasors of the two orthogonal polarization components. It is a complex number.

$$\tilde{r}_L = r_L e^{\delta_L} = \frac{\tilde{E}_y}{\tilde{E}_x} = \frac{E_y e^{j\delta_L}}{E_x} \quad \text{or} \quad \tilde{r}_L = \frac{\tilde{E}_V}{\tilde{E}_H} \tag{5.18}$$

Point of interest: In the case of circular polarization, the polarization ratio is defined as:

$$\tilde{r}_C = r_C e^{j\delta_C} = \frac{\tilde{E}_R}{\tilde{E}_L} \tag{5.19}$$

The circular polarization ratio $\tilde{r}_C$ is of particular interest since the axial ratio of the polarization ellipse *AR* can be expressed as:

$$AR = \frac{r_C + 1}{r_C - 1} \tag{5.20}$$

Besides, its tilt angle is:

$$\tau = \frac{\delta_C}{2} \tag{5.21}$$

Comparing (5.10) and (5.21) readily shows the relation between the phase difference of the circular-polarization representation and the linear polarization ratio $\tilde{r}_L = r_L e^{j\delta_L}$:

$$\delta_C = \arctan\left(\frac{2r_L}{1 - r_L^2}\cos\delta_L\right) \tag{5.22}$$

One can calculate r_C from the linear polarization ratio $\tilde{r}_L$ making use of (5.11) and (5.20):

$$\frac{r_C + 1}{r_C - 1} = \sqrt{\frac{1 + r_L^2 + \sqrt{1 + r_L^4 + 2r_L^2\cos(2\delta_L)}}{1 + r_L^2 - \sqrt{1 + r_L^4 + 2r_L^2\cos(2\delta_L)}}} \tag{5.23}$$

Using (5.22) and (5.23) allows easy switching between the representation of the wave polarization in terms of linear and circular components.

5. Antenna polarization

The polarization of a radiated wave (polarization of a radiating antenna) at a specific point in the far zone is the polarization of the locally plane wave.

The polarization of a received wave (polarization of a receiving antenna) is the polarization of a plane wave, incident from a given direction, and having given power flux density, which results in maximum available power at the antenna terminals.

6. Polarization loss factor and polarization efficiency

Generally, the polarization of the receiving antenna is not the same as the polarization of the incident wave. This is called ***polarization mismatch***.

The polarization loss factor (PLF) characterizes the loss of EM power because of polarization mismatch.

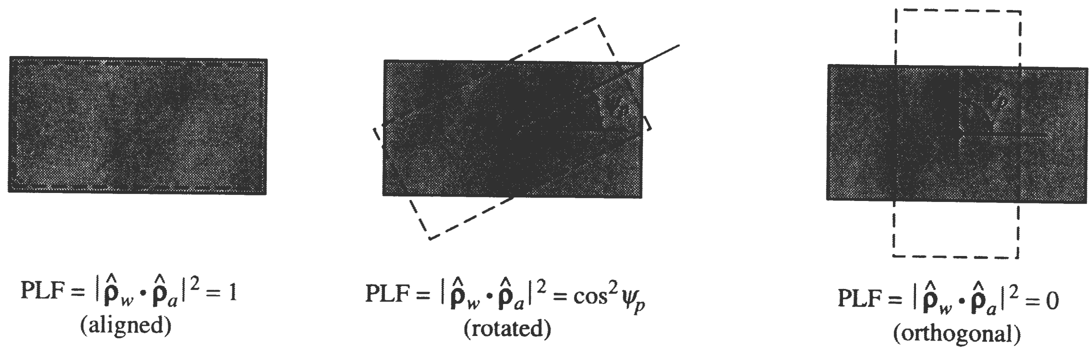
$$\text{PLF} = |\hat{\rho}_i \cdot \hat{\rho}_a|^2 \quad (5.24)$$

The above definition is based on the representation of the incident field and the antenna polarization by their polarization vectors. If the incident field is

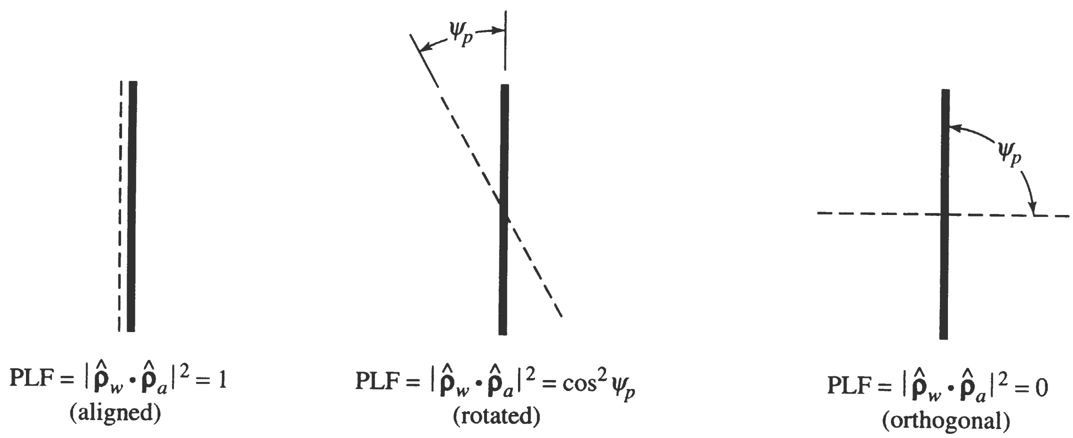
$$\vec{E}^i = E_m^i \hat{\rho}_i,$$

then the field of the same magnitude that would produce maximum received power at the antenna terminals is

$$\vec{E}_a = E_m^i \hat{\rho}_a.$$



(a) PLF for transmitting and receiving aperture antennas



If the antenna is polarization matched, then $\text{PLF}=1$, and there is no polarization power loss. If $\text{PLF}=0$, then the antenna is incapable of receiving the signal.

$$0 \le \text{PLF} \le 1 \tag{5.25}$$

The ***polarization efficiency*** has the same physical meaning as the PLF.

Examples

Example 5.1. The electric field of a linearly polarized EM wave is

$$\vec{E}^i = \hat{x}\cdot E_m(x,y)e^{-j\beta z}$$

It is incident upon a linearly polarized antenna whose polarization is:

$$\vec{E}_a = (\hat{x}+\hat{y})\cdot E(r,\theta,\varphi)$$

Find the PLF.

$$\text{PLF} = |\hat{x} \cdot \frac{1}{\sqrt{2}}(\hat{x} + \hat{y})|^2 = \frac{1}{2}$$

$$\text{PLF}_{[\text{dB}]} = 10\log_{10} 0.5 = -3$$

Example 5.2. A transmitting antenna produces a far-zone field, which is right-circularly polarized. This field impinges upon a receiving antenna, whose polarization (in transmitting mode) is also right-circular. Determine the PLF.

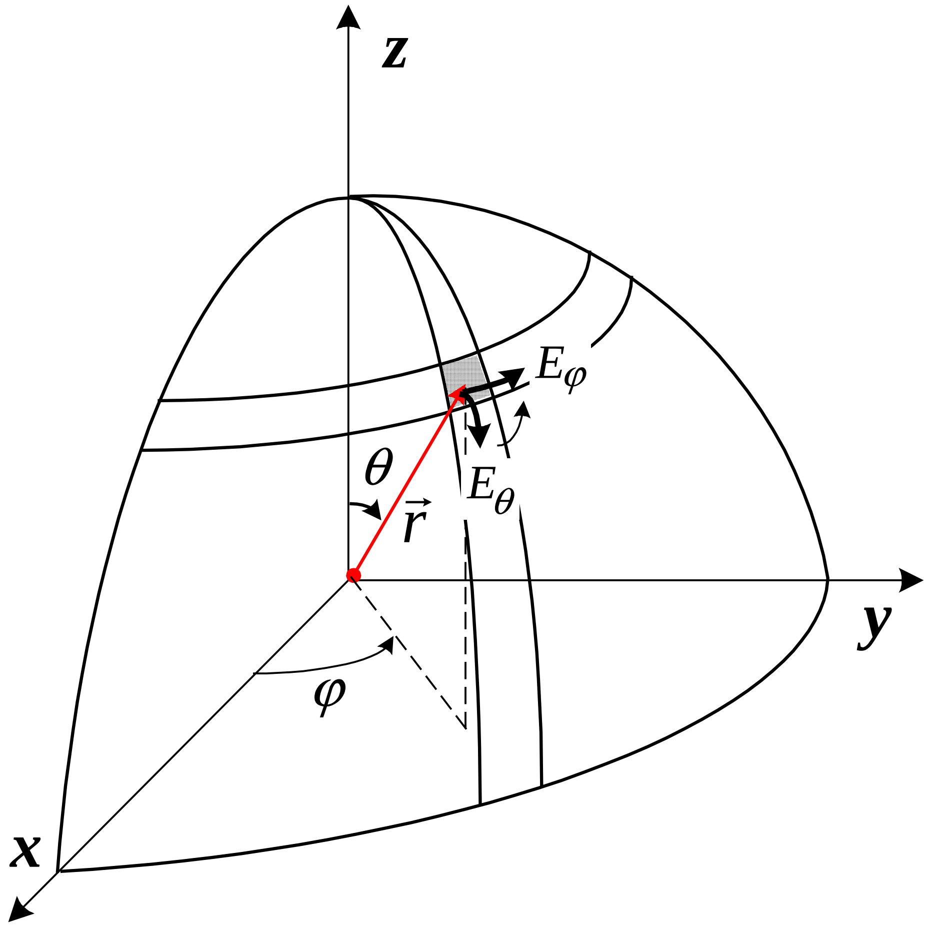
Both antennas (the transmitting one and the receiving one) are right-circularly polarized in transmitting mode. Let's assume that a transmitting antenna is located at the center of a spherical coordinate system. The far-zone field it would produce is described as:

$$\vec{E}^{\,far} = E_m \left[\hat{\theta} \cdot \cos\omega t + \hat{\varphi} \cdot \cos\left(\omega t - \pi/2\right) \right]$$

It is a right-circularly polarized field with respect to the outward radial direction. Its polarization vector is:

$$\hat{\rho} = \frac{\hat{\theta} - j\hat{\varphi}}{\sqrt{2}}$$

According to the definitions in Section 4, this is exactly the polarization vector of a transmitting antenna.



This same field $\vec{E}^{far}$ is incident upon a receiving antenna, which has the polarization vector $\hat{\rho}_a = \dfrac{\hat{\theta}_a - j\hat{\varphi}_a}{\sqrt{2}}$ in its own coordinate system $(r_a, \theta_a, \varphi_a)$. However, this field propagates along $-\vec{r}_a$ in the $(r_a, \theta_a, \varphi_a)$ coordinate system, and, therefore, its polarization vector becomes:

$$\hat{\rho}_i = \frac{\hat{\theta}_a + j\hat{\varphi}_a}{\sqrt{2}}$$

The PLF is calculated as:

$$\text{PLF} = |\hat{\rho}_i \cdot \hat{\rho}_a|^2 = \frac{|(\hat{\theta}_a + j\hat{\varphi}_a)(\hat{\theta}_a - j\hat{\varphi}_a)|^2}{4} = 1, \ \text{PLF}_{[\text{dB}]} = 10\log_{10} 1 = 0$$

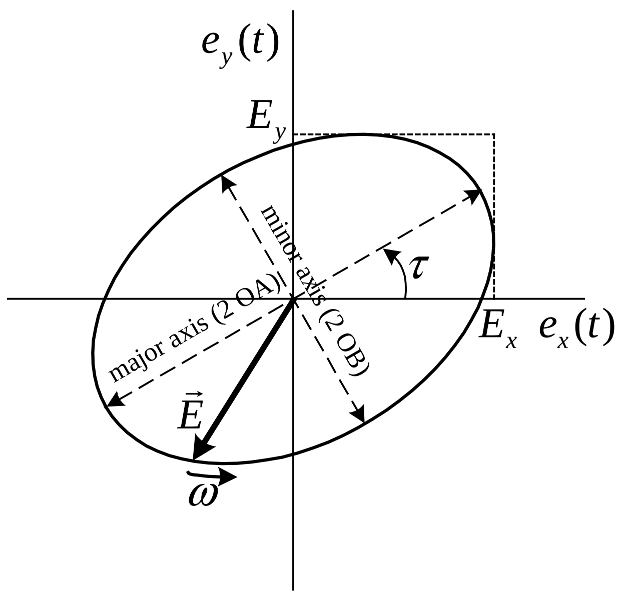
There are no polarization losses.

Exercise: Show that an antenna of right-circular polarization (in transmitting mode) cannot receive left-circularly polarized incident wave (or a wave emitted by a left-circularly polarized antenna).

Appendix I

Find the tilt angle τ, the length of the major axis OA, and the length of the minor axis OB of the ellipse described by the equation:

$$\sin^2\delta = \left[\frac{e_x(t)}{E_x}\right]^2 - 2\left[\frac{e_x(t)}{E_x}\right]\left[\frac{e_y(t)}{E_y}\right]\cos\delta + \left[\frac{e_y(t)}{E_y}\right]^2 \quad \text{(A-1)}$$



Equation (A-1) can be written as:

$$a \cdot x^2 - b \cdot xy + c \cdot y^2 = 1, \quad \text{(A-2)}$$

where:

$x = e_x(t)$ and $y = e_y(t)$ are the coordinates of a point of the ellipse centered in the $x - y$ plane;

$a = \dfrac{1}{E_x^2 \sin^2\delta}$;

$b = \dfrac{2\cos\delta}{E_x E_y \sin^2\delta}$;

$$c = \frac{1}{E_y^2 \sin^2 \delta}.$$

After dividing both sides of (A-2) by (xy), one obtains

$$a\frac{x}{y} - b + c\frac{y}{x} = \frac{1}{xy} \quad \text{(A-3)}$$

Introducing $\xi = \frac{y}{x} = \frac{e_y(t)}{e_x(t)}$, one obtains that

$$x^2 = \frac{1}{c\xi^2 - b\xi + a}$$

$$\Rightarrow \rho^2(\xi) = x^2 + y^2 = x^2(1+\xi^2) = \frac{(1+\xi^2)}{c\xi^2 - b\xi + a} \quad \text{(A-4)}$$

Here, ρ is the distance of the ellipse point from the center of the coordinate system. We want to know at what ξ values the maximum and the minimum of ρ occur. This will produce the tilt angle τ. We also want to know what are the values of $\rho_{\max}$ and $\rho_{\min}$. Then, we have to solve

$$\frac{d(\rho^2)}{d\xi} = 0, \text{ or}$$

$$\xi^2 - \frac{2(a-c)}{b}\xi - 1 = 0 \quad \text{(A-5)}$$

(A-5) is better solved for the angle α, such that

$$\xi = \tan\alpha = \frac{y}{x}; \quad \alpha = \frac{\pi}{2} - \tau \quad \text{(A-6)}$$

Substituting (A-6) in (A-5) yields:

$$\left(\frac{\sin\alpha}{\cos\alpha}\right)^2 - 2C\left(\frac{\sin\alpha}{\cos\alpha}\right) - 1 = 0 \quad \Big| \times \cos^2\alpha \quad \text{(A-7)}$$

where $C = \frac{a-c}{b} = \frac{E_y^2 - E_x^2}{2E_xE_y\cos\delta}$.

The solution of (A-7) is:

$$\alpha_1 = \frac{1}{2}\arctan\left(\frac{2E_xE_y\cos\delta}{E_x^2 - E_y^2}\right); \quad \alpha_2 = \alpha_1 + \frac{\pi}{2}$$
$$\Rightarrow \tau_{1,2} = \alpha_{1,2} \tag{A-8}$$

Substituting α_1 and α_2 back in ρ yields the expressions for OA and OB.