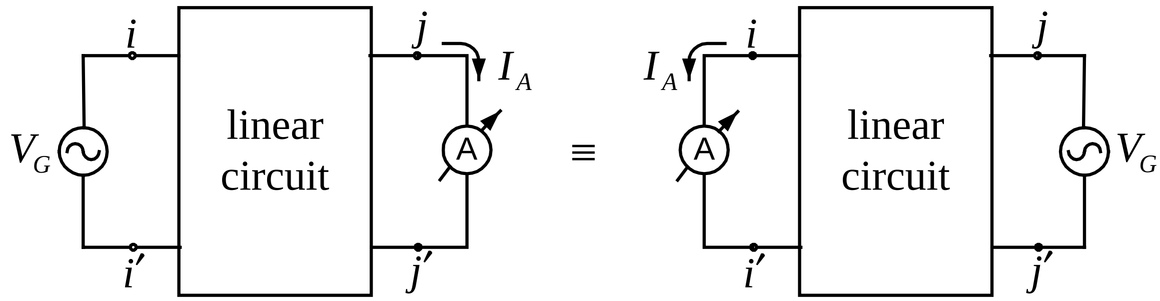
LECTURE 9: Cylindrical Antennas – Classical Theoretical Models

(*Reciprocity theorem. Self-impedance of a dipole using the induced emf method. Pocklington's equation. Hallén's equation.*)

1. Reciprocity theorem for antennas

1.1. Reciprocity theorem in circuit theory

If a voltage (current) generator is placed between any pair of nodes of a linear circuit, and a current (voltage) reaction is measured between any other pair of nodes, the interchange of the generator's and the measurement's locations would lead to the same measurements results.



$$\boxed{\frac{V_i}{I_j} = \frac{V_j}{I_i} \quad \text{or} \quad Z_{ji} = Z_{ij}} \tag{9.1}$$

1.2. Reciprocity theorem in EM field theory (Lorentz' reciprocity theorem)

Consider a volume $V_{[S]}$ bounded by the surface S, where two pairs of sources exist: $(\vec{J}_1, \vec{M}_1)$ and $(\vec{J}_2, \vec{M}_2)$. We shall denote the fields associated with the $(\vec{J}_1, \vec{M}_1)$ sources as $(\vec{E}_1, \vec{H}_1)$, and the fields generated by $(\vec{J}_2, \vec{M}_2)$ as $(\vec{E}_2, \vec{H}_2)$.

$$\left|\begin{array}{ll}\nabla\times\vec{E}_1=-j\omega\mu\vec{H}_1-\vec{M}_1 & /\cdot\vec{H}_2\\ \nabla\times\vec{H}_1=j\omega\varepsilon\vec{E}_1+\vec{J}_1 & /\cdot\vec{E}_2\end{array}\right. \tag{9.2}$$

$$\left|\begin{array}{ll}\nabla\times\vec{E}_2=-j\omega\mu\vec{H}_2-\vec{M}_2 & /\cdot\vec{H}_1\\ \nabla\times\vec{H}_2=j\omega\varepsilon\vec{E}_2+\vec{J}_2 & /\cdot\vec{E}_1\end{array}\right. \tag{9.3}$$

The vector identity

$$\nabla\cdot(\vec{E}_1\times\vec{H}_2-\vec{E}_2\times\vec{H}_1)=\vec{H}_2\cdot\nabla\times\vec{E}_1-\vec{E}_1\cdot\nabla\times\vec{H}_2-\vec{H}_1\cdot\nabla\times\vec{E}_2+\vec{E}_2\cdot\nabla\times\vec{H}_1$$

is used to obtain:

$$\nabla\cdot(\vec{E}_1\times\vec{H}_2-\vec{E}_2\times\vec{H}_1)=-\vec{E}_1\cdot\vec{J}_2+\vec{H}_1\cdot\vec{M}_2+\vec{E}_2\cdot\vec{J}_1-\vec{H}_2\cdot\vec{M}_1 \tag{9.4}$$

Equation (9.4) can be written in its integral form as:

$$\oint\!\!\oint_S(\vec{E}_1\times\vec{H}_2-\vec{E}_2\times\vec{H}_1)d\vec{s}=\iiint_{V_{[S]}}(-\vec{E}_1\cdot\vec{J}_2+\vec{H}_1\cdot\vec{M}_2+\vec{E}_2\cdot\vec{J}_1-\vec{H}_2\cdot\vec{M}_1)dv \tag{9.5}$$

Equations (9.4) and (9.5) represent the general form of the Lorentz' reciprocity theorem in differential and in integral form, respectively.

One special case of the reciprocity theorem is of fundamental importance to antenna theory, namely its application to unbounded (open) problems. In this case, the surface S is of infinite radius. Therefore, the fields integrated over the surface S are far-zone fields, which means that the left-hand side of (9.5) vanishes:

$$\oint\!\!\oint_S(\vec{E}_1\times\vec{H}_2-\vec{E}_2\times\vec{H}_1)d\vec{s}=\oint\!\!\oint_S\left(\frac{|\vec{E}_1||\vec{E}_2|}{\eta}\cos\gamma-\frac{|\vec{E}_1||\vec{E}_2|}{\eta}\cos\gamma\right)d\vec{s}=0 \tag{9.6}$$

Here, γ is the angle between the polarization vectors of both fields, $\vec{E}_1$ and $\vec{E}_2$. It follows that in the case of open problems the reciprocity theorem reduces to:

$$\iiint_{V_{[S]}}(\vec{E}_1\cdot\vec{J}_2-\vec{H}_1\cdot\vec{M}_2)dv=\iiint_{V_{[S]}}(\vec{E}_2\cdot\vec{J}_1-\vec{H}_2\cdot\vec{M}_1)dv \tag{9.7}$$

Each of the integrals in (9.7) can be interpreted as *coupling energy* between the field produced by some sources and another set of sources, which generate another field. The quantity

$$\langle 1,2\rangle=\iiint_{V_{[S]}}(\vec{E}_1\cdot\vec{J}_2-\vec{H}_1\cdot\vec{M}_2)dv$$

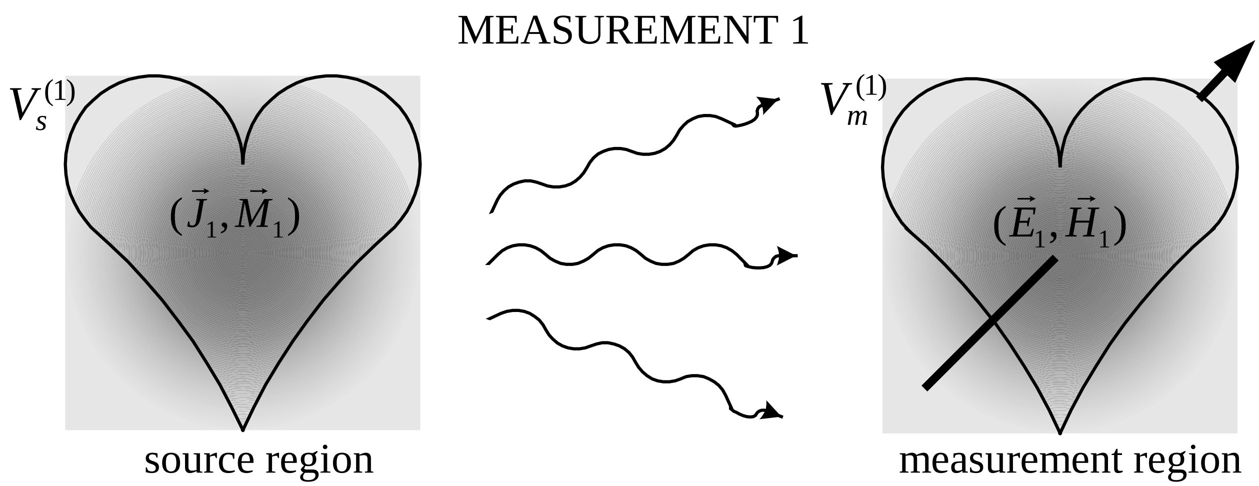
is called ***reaction*** of the field $(\vec{E}_1, \vec{H}_1)$ to the sources $(\vec{J}_2, \vec{M}_2)$. Similarly, the quantity

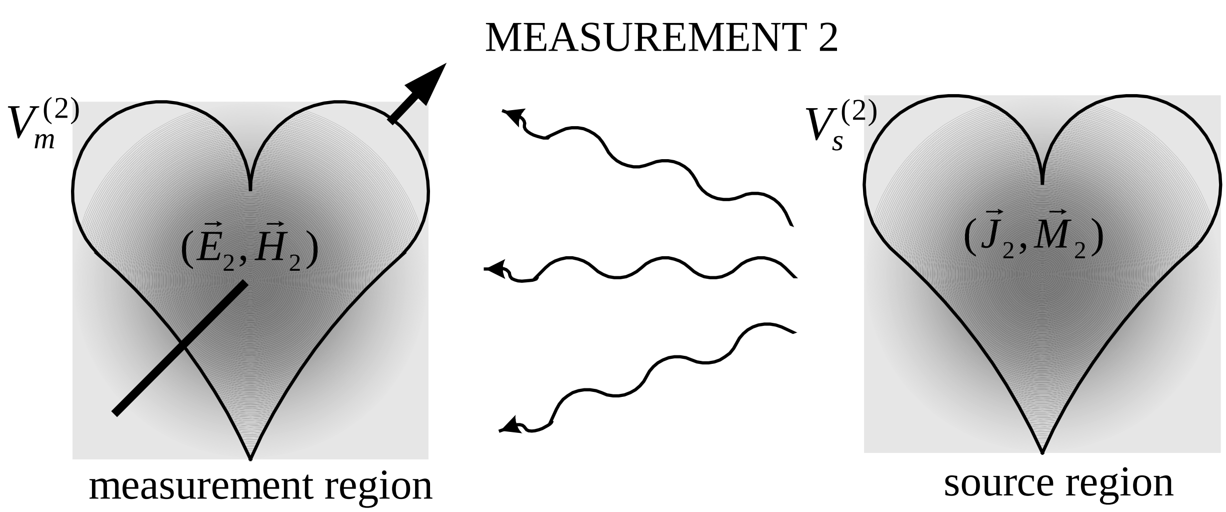
$$\langle 2,1 \rangle = \iiint\limits_{V_{[S]}} (\vec{E}_2 \cdot \vec{J}_1 - \vec{H}_2 \cdot \vec{M}_1) dv$$

is the ***reaction*** of the field $(\vec{E}_2, \vec{H}_2)$ to the sources $(\vec{J}_1, \vec{M}_1)$. Equation (9.7) can be briefly written as

$$\boxed{\langle 1,2 \rangle = \langle 2,1 \rangle} \tag{9.8}$$

The Lorentz' reciprocity theorem is the most general form of reciprocity in linear electromagnetic systems. Circuit theory reciprocity is a special case of lumped element sources and reaction (local voltage or current measurements). To illustrate the above statement, consider the following experiment:





Assume that the sources for both measurements have identical local amplitude-phase distribution; that is: $V_s^{(1)} = V_s^{(2)} = V_s$, and $\vec{J}_1 = \vec{J}_2$, $\vec{M}_1 = \vec{M}_2$ everywhere in the source volume. Assume also that the measurements in both cases are carried out in a volume of the same shape and dimensions $V_m^{(1)} = V_m^{(2)} = V_m$. According to (9.7)

$$\iiint\limits_{V_{[S]}} (\vec{E}_1 \cdot \vec{J} - \vec{H}_1 \cdot \vec{M}) dv = \iiint\limits_{V_{[S]}} (\vec{E}_2 \cdot \vec{J} - \vec{H}_2 \cdot \vec{M}) dv \qquad (9.9)$$

$\Rightarrow \vec{E}_1 = \vec{E}_2$ and $\vec{H}_1 = \vec{H}_2$. *Reaction (measurement data) is insensitive to the interchange of source and measurement locations*. This is the same principle that was postulated as reciprocity in circuit theory (see Section 1.1). Only that we consider volumes instead of nodes, and field vectors instead of voltages and currents.

The general reciprocity theorem can be postulated also as: *any network constructed of linear isotropic matter has a symmetrical impedance matrix*. This "network" can be two antennas and the space between them.

1.3. Reciprocity in antenna theory

According to the reciprocity theorem, if antenna #1 is a transmitting antenna and antenna #2 is the receiving one, ***the ratio of transmitted to received power*** P_t / P_r ***will not change*** if antenna #1 becomes the receiving antenna and antenna #2 becomes the transmitting one. It should be reiterated that the reciprocity theorem holds only if the whole system (antennas + propagation environment) is isotropic and linear.

Let us assume that antennas #1 and #2 are matched to their feed networks. Then the power fed to antenna #1 (the transmitting one) will be:

$$P_1 = \frac{|V_g|^2}{8R_{A_1}}, \qquad (9.10)$$

where $R_{A_1} = \mathrm{Re}\{Z_{A_1}\} = R_{r_1} + R_{l_1}$ is the resistance of antenna #1. If the transfer admittance of the combined network consisting of antenna #1, free space and antenna #2, is Y_{21}, then the current at the antenna #2 load is $(V_g Y_{21})$. The power delivered to the load is:

$$P_2 = \frac{1}{2} R_{A_2} |V_g|^2 |Y_{21}|^2 = \frac{1}{2} R_{L_2} |V_g|^2 |Y_{21}|^2 \qquad (9.11)$$

$$\Rightarrow \frac{P_r}{P_t} = \frac{P_2}{P_1} = 4 R_{A_1} R_{A_2} |Y_{21}|^2 \qquad (9.12)$$

In a similar manner, it can be shown that the ratio P_r / P_t when antenna #2 transmits and antenna #1 receives, is:

$$\frac{P_r}{P_t} = \frac{P_1}{P_2} = 4 R_{A_1} R_{A_2} |Y_{12}|^2 \qquad (9.13)$$

Since $Y_{21} = Y_{12}$, it follows that ***the ratio of received to transmitted power does not change if the antennas interchange receiving with transmitting mode***.

Reciprocity of the radiation pattern

The radiation pattern is the same in receiving and in transmitting modes if the materials used for the construction of the antenna and the transmission lines are linear and the medium of wave propagation is linear. Nonlinear devices such as diodes and transistors make the antenna system nonlinear, therefore, nonreciprocal.

The two-port model of a measurement system is:

$$\left| \begin{aligned} V_1 &= Z_{11} I_1 + Z_{12} I_2 \\ V_2 &= Z_{21} I_1 + Z_{22} I_2 \end{aligned} \right. \qquad (9.14)$$

Here:

Z_{11} - self-impedance of antenna #1

Z_{22} - self-impedance of antenna #2

Z_{12}, Z_{21} - mutual impedances.

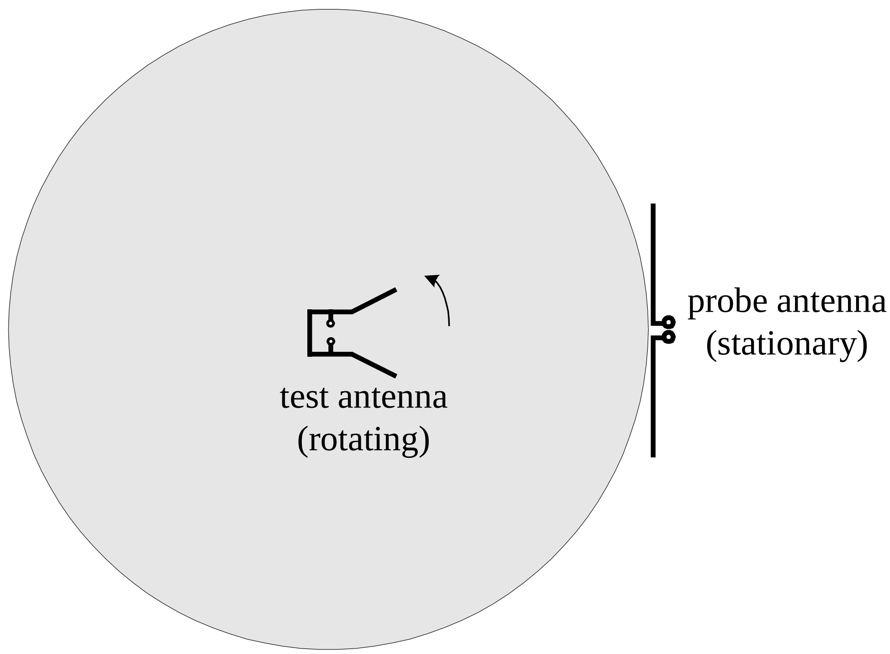
The field pattern is measured as the open-circuit voltage of the receiving antenna. If antenna #1 transmits, then the voltage

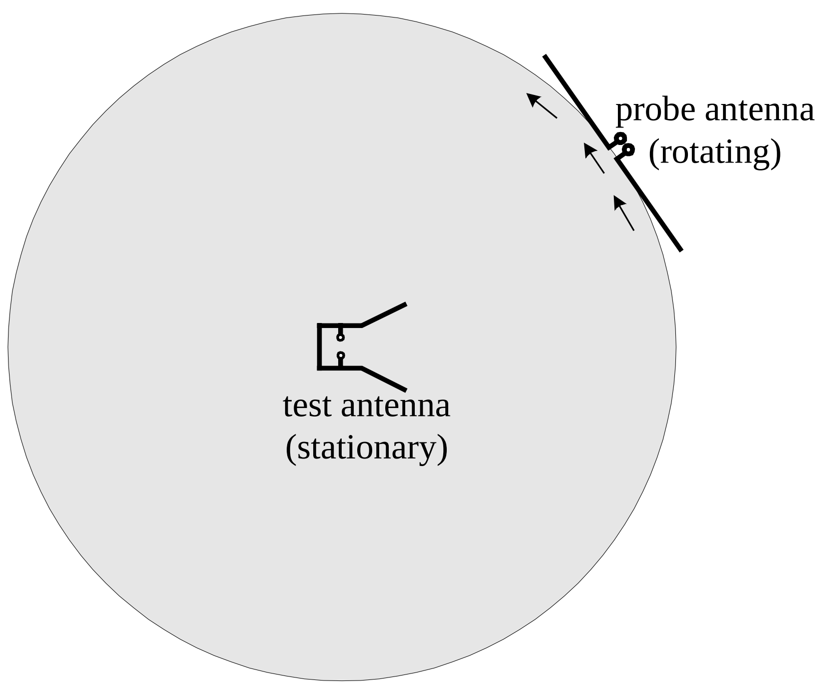
$$V_{oc_2} = Z_{21} I_1 \quad _{/ I_2 = 0} \qquad (9.15)$$

is measured. If antenna #2 transmits, then the voltage

$$V_{oc_1} = Z_{12} I_2 \quad _{/ I_1 = 0} \qquad (9.16)$$

is measured. The excitation currents I_1 and I_2 do not depend on the direction and cannot influence the shape of the pattern $V_{oc}(\theta, \varphi)$. The pattern depends only on the mutual impedance associated with a given direction $Z_{12}(\theta,\varphi) = Z_{21}(\theta,\varphi)$. It is now obvious that the pattern would not depend on whether the antenna under test receives and the probe antenna transmits, or vice versa. It also does not matter whether the antenna under test rotates and the probe antenna is stationary, or vice versa.

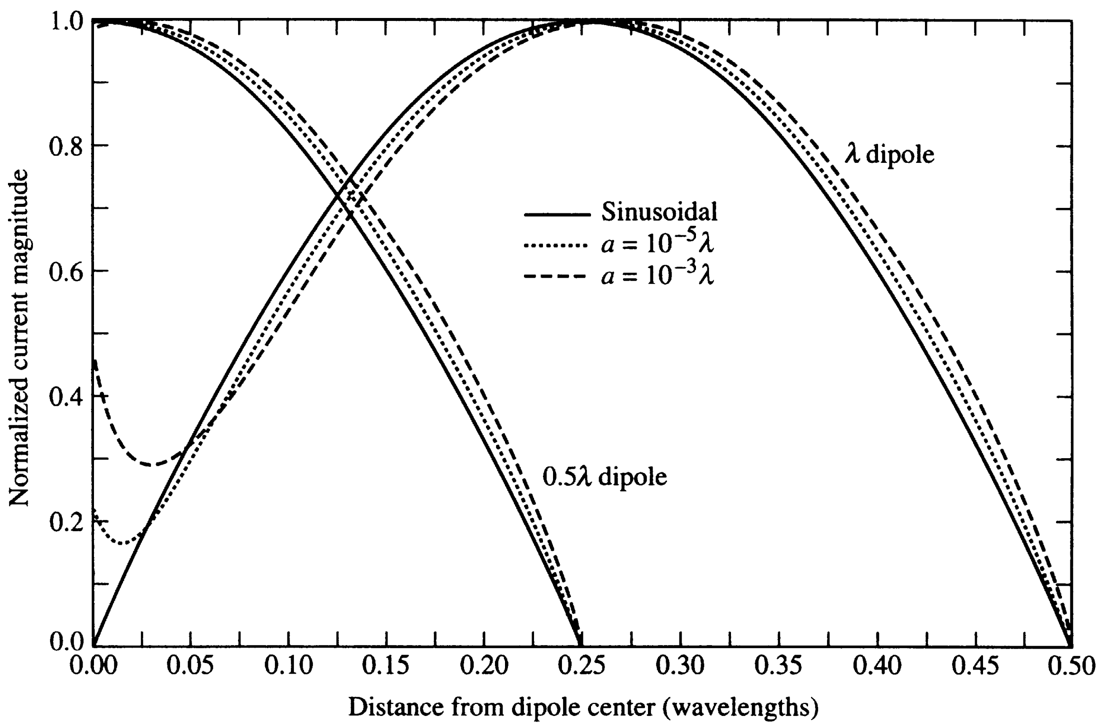




2. Self impedance of a dipole using the induced emf method

The induced emf method was developed by Carter[1] in 1932, when powerful computers were not available and analytical (closed-form) solutions were very much needed to calculate the self-impedance of wire antennas. The method was later extended to calculate mutual impedances of multiple wires. The emf method is restricted to straight parallel wires.

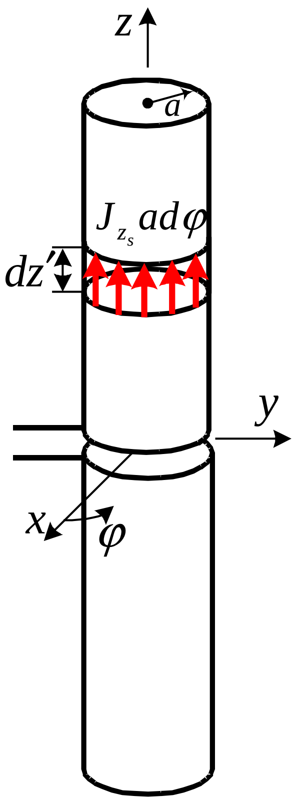
Measurements indicate that the current distribution on thin dipoles is nearly sinusoidal (except at the current minima). The induced emf method assumes this type of idealized distribution. It results in satisfactory accuracy for dipoles with length-diameter ratios as small as 100, provided the terminals are at the current maximum.



Let's assume that the feed is at the current maximum I_m. Using the reciprocity principle one can show that

$$V_m I_m = -\int_{-l/2}^{l/2} I_z(\rho = a, z = z')E_z(\rho = a, z = z')dz' \qquad (9.17)$$

[1] P.S. Carter, "Circuit relations in radiating systems and applications to antenna problems," *Proc. IRE*, **20**, pp.1004-1041, June 1932.



V_m is the voltage applied at the antenna feed. $I_z(\rho, z)$ is the resulting current at the wire. I_z produces the field $E_z(\rho, z)$, which can be calculated using the VP $\vec{A}$, as if there were no conductor backing the currents. It is not zero at the conducting surface of the wire, which implies the existence of an induced field E_{z_i} such that $E_{z_i} = -E_z$. The induced field E_{z_i} would produce the current I_m at the antenna input.

The input (self) impedance of the antenna is:

$$Z_m = \frac{V_m}{I_m} \tag{9.18}$$

It follows that:

$$Z_m = -\frac{1}{I_m^2} \int_{-l/2}^{l/2} I_z(\rho = a, z = z') E_z(\rho = a, z = z') dz' \tag{9.19}$$

Note that in (9.17) and in (9.19) it is assumed that currents are concentrated at the wire's surface (which is practically true for all copper and aluminum dipoles), and that the currents and the resulting E_z field are not dependent on the angle φ. The latter assumption is only true for thin wires, as mentioned above. Another assumption made is that the current distribution along the wire induced by the applied voltage V_m is sinusoidal:

$$I(z') = \begin{cases} I_m \sin\left[\beta\left(\dfrac{l}{2} - z'\right)\right], & 0 \le z' \le l/2 \\ I_m \sin\left[\beta\left(\dfrac{l}{2} + z'\right)\right], & -l/2 \le z' \le 0 \end{cases} \tag{9.20}$$

So far, we have obtained only the far-field components of the field generated by the current in (9.20) (see Lecture 6). However, when the input resistance and reactance are to be found, the total near fields have to be known. In our case, we are particularly interested in E_z, which is the field produced by $I(z')$ as if there is no conductor surface present. We shall use cylindrical coordinates (ρ, φ, z) to describe the location of the integration and the observation points. The electric field can be expressed in terms of the VP $\vec{A}$ and the scalar potential ϕ (see Lecture 2).

$$\vec{E} = -\nabla\phi - j\omega\vec{A} \tag{9.21}$$

$$\Rightarrow E_z = -\frac{\partial\phi}{\partial z} - j\omega A_z \tag{9.22}$$

The VP $\vec{A}$ is the superposition of the retarded current potentials:

$$\vec{A} = \frac{\mu}{4\pi}\int_{-l/2}^{l/2} I_z(\rho = a, z = z')\frac{e^{-j\beta R}}{R}dz' \tag{9.23}$$

The scalar potential is the superposition of the retarded charge potentials:

$$\phi = \frac{1}{4\pi\varepsilon}\int_{-l/2}^{l/2} q_l(\rho = a, z = z')\frac{e^{-j\beta R}}{R}dz' \tag{9.24}$$

Here, q_l stands for linear charge density (C/m). Knowing that the current depends only on z, the continuity relation is written as:

$$j\omega q_l = -\frac{\partial I_z}{\partial z'} \tag{9.25}$$

$$\Rightarrow q_l(z') = \begin{cases} -j\dfrac{I_m}{c}\cos\left[\beta\left(\dfrac{l}{2} - z'\right)\right], & 0 \le z' \le l/2 \\ +j\dfrac{I_m}{c}\cos\left[\beta\left(\dfrac{l}{2} + z'\right)\right], & -l/2 \le z' \le 0 \end{cases} \tag{9.26}$$

where $c = \omega/\beta$ is the speed of light. Now, we can write the expressions for $\vec{A}$ and ϕ.

$$A_z = \frac{\mu}{4\pi}I_m\left\{\int_{-l/2}^{0}\sin\left[\beta\left(\frac{l}{2}+z'\right)\right]\frac{e^{-j\beta R}}{R}dz' + \int_{0}^{l/2}\sin\left[\beta\left(\frac{l}{2}-z'\right)\right]\frac{e^{-j\beta R}}{R}dz'\right\} \tag{9.27}$$

$$\phi = j\frac{\eta I_m}{4\pi}\left\{-\int_{-l/2}^{0}\cos\left[\beta\left(\frac{l}{2}+z'\right)\right]\frac{e^{-j\beta R}}{R}dz' + \int_{0}^{l/2}\cos\left[\beta\left(\frac{l}{2}-z'\right)\right]\frac{e^{-j\beta R}}{R}dz'\right\} \tag{9.28}$$

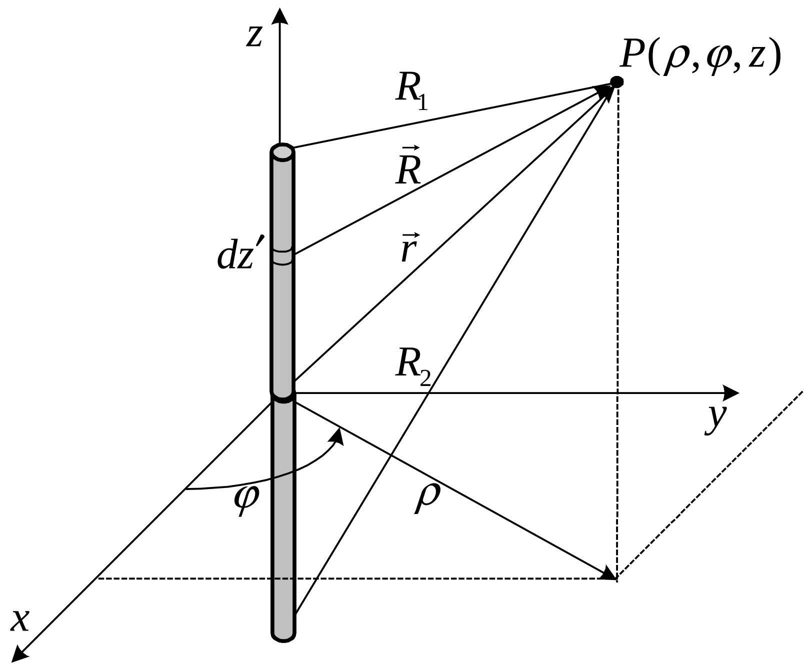
Here, $\eta = \sqrt{\mu/\varepsilon}$ is the intrinsic impedance of the medium ($\approx 120\pi$ Ω in vacuum).

The distance between integration and observation point is assumed:

$$R = \sqrt{\rho^2 + (z - z')^2}\,, \tag{9.29}$$

which is actually an approximation of:

$$R = \sqrt{(\rho - a)^2 + (z - z')^2} \tag{9.30}$$



Equation (9.29) is substituted in (9.27) and (9.28). The resulting equations for $\vec{A}$ and ϕ are modified making use of Moivre's formulas:

$$\begin{aligned} \cos x &= \frac{1}{2}\left(e^{jx} + e^{-jx}\right) \\ \sin x &= \frac{1}{2j}\left(e^{jx} - e^{-jx}\right) \end{aligned} \tag{9.31}$$

Then, the equations of $\vec{A}$ and ϕ are substituted in (9.22) to derive the expression for E_z. This is a rather lengthy derivation, and we shall give the final result only:

$$E_z = -j\frac{\eta I_m}{4\pi}\left[\frac{e^{-j\beta R_1}}{R_1} + \frac{e^{-j\beta R_2}}{R_2} - 2\cos\left(\frac{\beta l}{2}\right)\frac{e^{-j\beta r}}{r}\right] \tag{9.32}$$

The final goal of this discussion is to find the self-impedance (9.19) of the dipole. Equation (9.19) can be written as:

$$Z_m = -\frac{1}{I_m} \int_{-l/2}^{l/2} \sin\left[\beta\left(\frac{l}{2} - |z'|\right)\right] E_z(\rho = a, z = z')dz' \tag{9.33}$$

Substituting (9.33) in (9.32) produces the following results for the real and the imaginary part of Z_{in}:

$$R_m = R_r = \frac{\eta}{2\pi}\left\{C + \ln(\beta l) - C_i(\beta l) + \frac{1}{2}\sin(\beta l)\left[S_i(2\beta l) - 2S_i(\beta l)\right] + \right.$$
$$\left. + \frac{1}{2}\cos(\beta l)\left[C + \ln(\beta l/2) + C_i(2\beta l) - 2C_i(\beta l)\right]\right\} \tag{9.34}$$

$$X_m = \frac{\eta}{4\pi}\left\{2S_i(\beta l) - \cos(\beta l)\left[S_i(2\beta l) - 2S_i(\beta l)\right] + \right.$$
$$\left. + \sin(\beta l)\left[C_i(2\beta l) - 2C_i(\beta l) + C_i\left\{\frac{2\beta a^2}{l}\right\}\right]\right\} \tag{9.35}$$

The equations above refer to a feed point, which coincides with the maximum of the current distribution. The dipole is normally fed at the center. However, the current has its maximum at the center only if

$$l = (2k+1)\frac{\lambda}{2}, \quad k = 0, 1, \ldots \tag{9.36}$$

If the length of the antenna is other than that in (9.36), the input impedance will differ from that in (9.34) and (9.35). The relation between the input resistance at the maximum-current point and the resistance at the centered-feed point for any dipole length was already found in Lecture 6 (see equation 6.34). The same relation holds for the reactances, too:

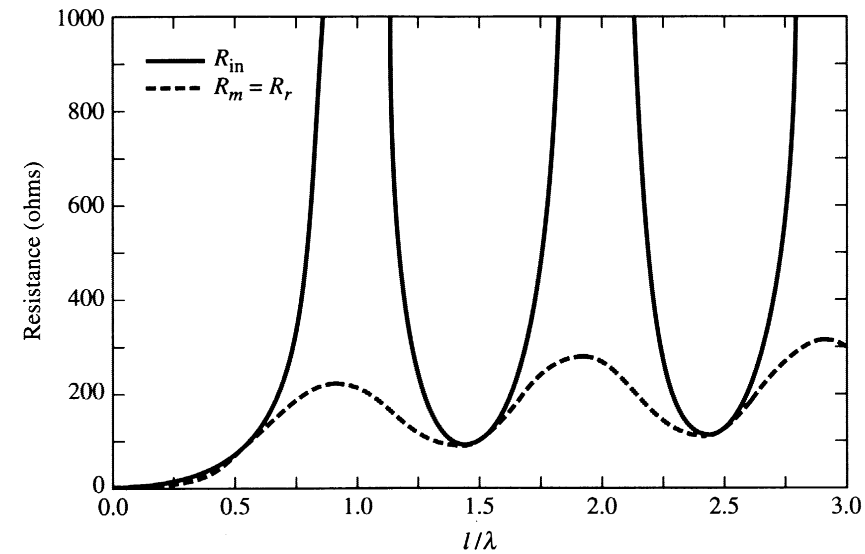
$$R_{in} = \left(\frac{I_m}{I_{in}}\right)^2 R_r = \frac{R_r}{\sin^2\left(\frac{\beta l}{2}\right)} \tag{9.37}$$

$$X_{in} = \left(\frac{I_m}{I_{in}}\right)^2 X_m = \frac{X_m}{\sin^2\left(\frac{\beta l}{2}\right)} \tag{9.38}$$

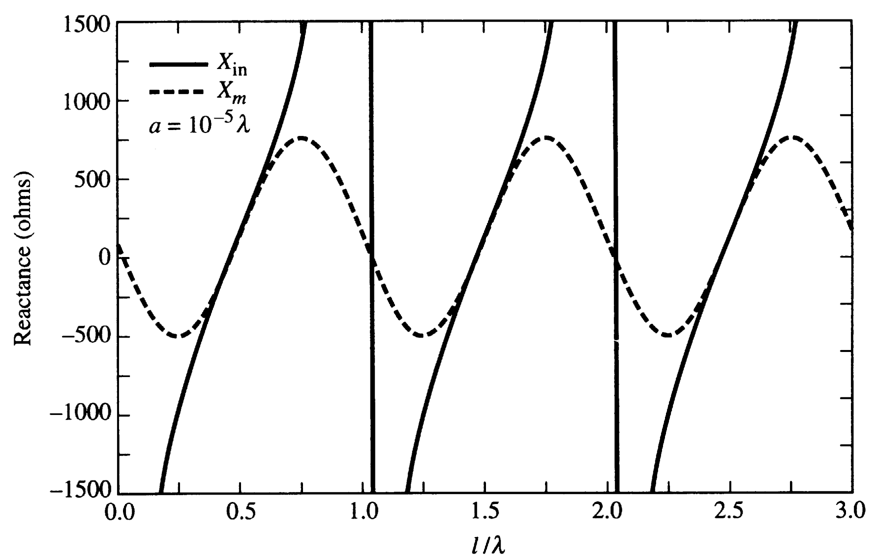
For a small dipole, the input reactance can be approximated by:

$$X_{in} = X_m \simeq -120\frac{[\ln(l/a)-1]}{\tan(\beta l)} \tag{9.39}$$

The results produced by (9.37) and (9.38) for different ratios l/λ are given in the plots below.

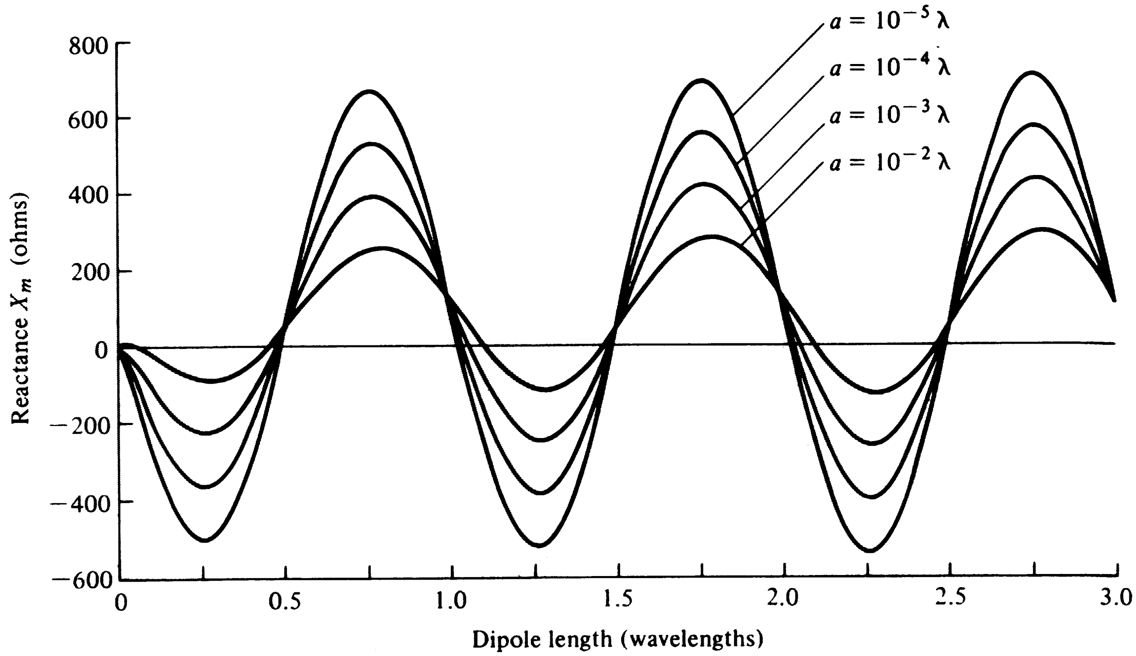


(a) Resistance



(b) Reactance

Reactance of a thin dipole (emf method) for different wire radii a:



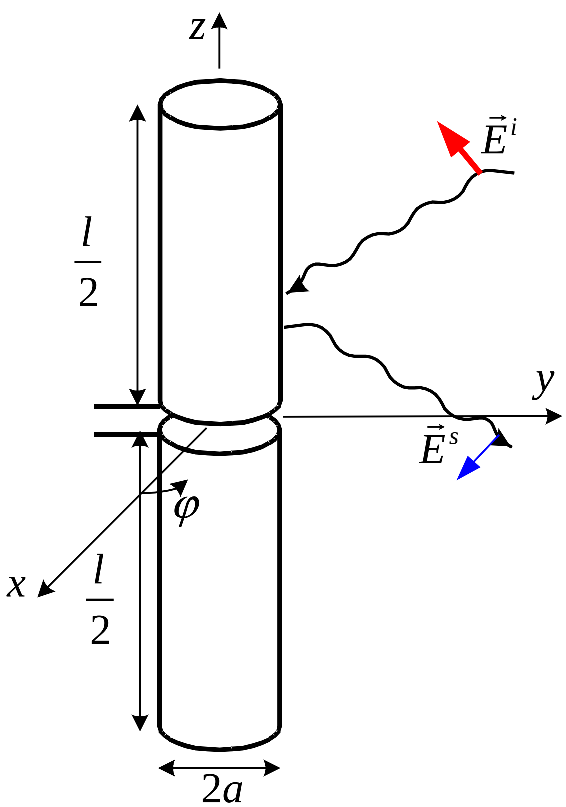
Note that:

- the reactance does not depend on the radius a, when the dipole length is a multiple of a half-wavelength ($l = n\lambda/2$), as follows from (9.35);
- the resistance does not depend on a according to the assumptions made in the emf method (see equation (9.34)).

3. Pocklington's equation

The assumption of a sinusoidal current distribution along the dipole is considered accurate enough for wire diameters $d < 0.05\lambda$. Generally speaking, the current distribution is not sinusoidal in the case of thicker wires. The currents must be computed using some general numerical approach. Below, we shall introduce two integral equations, which can produce the current distribution on any wire antenna of finite diameter. These equations are classics in wire antenna theory. We shall not discuss in detail their numerical solution, which is somewhat beyond the subject of this course.

To derive Pocklington's equation, the concepts of incident and scattered field will be introduced first.



The incident wave is a wave produced by some sources. This wave would exist in the location of the scatterer, if the scatterer were not present. The scatterer though is present, and if it is a conducting body, it would require the vanishing of the electric field components tangential to its surface

$$\vec{E}_\tau^t = 0 \tag{9.40}$$

The vector $\vec{E}^t$ denotes the so-called total electric field. This means that as the non-zero incident field impinges upon the conducting scatterer, it induces on its surface currents $\vec{J}_s$, which in their turn produce a field, the scattered field $\vec{E}^s$. The scattered and the incident fields superimpose to form the total field:

$$\vec{E}^t = \vec{E}^i + \vec{E}^i \tag{9.41}$$

The scattered field is such that (9.40) is fulfilled, i.e.

$$\vec{E}_\tau^s = -\vec{E}_\tau^i \tag{9.42}$$

Any irradiated object presenting a discontinuity in the wave's propagation path is a scatterer, and so is any receiving antenna. However, the above concepts hold for transmitting antennas, too. In the case of a wire dipole, the incident field exists only at the base of the dipole (in its feed gap).

In the case of cylindrical dipoles with excitation of cylindrical symmetry, the $\vec{E}$ field has no φ-component. The only tangential component is the z one. The boundary condition at the dipole's surface is:

$$E_z^s = -E^i_{z/\rho=a,-\frac{l}{2}<z<\frac{l}{2}} \tag{9.43}$$

The scattered field can be expressed in terms of $\vec{A}$ and ϕ, as it was already done in (9.22):

$$E_z^s = -\frac{\partial\phi}{\partial z} - j\omega A_z = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial z^2} - j\omega A_z \tag{9.44}$$

or

$$E_z^s = -j\frac{1}{\omega\mu\varepsilon}\left(\beta^2 A_z + \frac{\partial^2 A_z}{\partial z^2}\right) \tag{9.45}$$

We assume only z-components of the surface currents and no edge effects:

$$A_z(\rho,\varphi,z) = \frac{\mu}{4\pi}\int_{-l/2}^{l/2}\int_0^{2\pi} J_z \frac{e^{-j\beta R}}{R}\underbrace{a\,d\varphi' dz'}_{ds} \tag{9.46}$$

If the cylindrical symmetry of the dipole and the excitation are preserved, the current J_z does not depend on the azimuthal angle φ. It can be shown that the field created by a cylindrical sheet of surface currents J_z is equivalent to the field created by a current filament of current I_z:

$$2\pi a J_z = I_z \Rightarrow J_z(z') = \frac{1}{2\pi a} I_z(z') \tag{9.47}$$

Then, (9.46) reduces to:

$$A_z(\rho,\varphi,z) = \frac{\mu}{4\pi}\int_{-l/2}^{l/2}\frac{1}{2\pi a}\int_0^{2\pi} I_z(z')\frac{e^{-j\beta R}}{R}a\,d\varphi' dz' \tag{9.48}$$

The distance between observation and integration points is:

$$\begin{aligned} R &= \sqrt{(x-x')^2+(y-y')^2+(z-z')^2} = \\ &= \sqrt{\rho^2+a^2-2\rho a\cos(\varphi-\varphi')+(z-z')^2} \end{aligned} \tag{9.49}$$

The cylindrical geometry of the problem implies the cylindrical symmetry of the observed fields, i.e. $\vec{A}$ does not depend on φ and one can assume that $\varphi = 0$. Besides, we are interested in the scattered field produced by this equivalent current at the dipole's surface, i.e. the observation point is at $\rho = a$. Then,

$$A_z(a,0,z) = \mu \int_{-l/2}^{l/2} I_z(z') \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-j\beta R}}{4\pi R} d\varphi' \right) dz' = \mu \int_{-l/2}^{l/2} I_z(z') G\left(z,z'\right) dz', \tag{9.50}$$

where

$$G\left(z,z'\right) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-j\beta R}}{4\pi R} d\varphi' \tag{9.51}$$

$$R_{(\rho=a,\varphi=0)} = \sqrt{4a^2 \sin^2\left(\frac{\varphi'}{2}\right) + (z-z')^2} \tag{9.52}$$

Substituting (9.50) in (9.45) yields:

$$E_z^s(\rho=a) = -j\frac{1}{\omega\varepsilon}\left(\beta^2+\frac{d^2}{dz^2}\right)\int_{-l/2}^{l/2} I_z(z')G(z,z')dz' \tag{9.53}$$

Imposing the boundary condition (9.43) on the field in (9.53) leads to:

$$\left(\beta^2+\frac{d^2}{dz^2}\right)\int_{-l/2}^{l/2} I_z(z')G(z,z')dz' = -j\omega\varepsilon E_z^i(\rho=a) \tag{9.54}$$

Since I_z is the source, which does not depend on z, (9.54) can be rewritten as:

$$\int_{-l/2}^{l/2} I_z(z')\left(\beta^2 G(z,z') + \frac{d^2 G(z,z')}{dz^2}\right) dz' = -j\omega\varepsilon E_z^i(\rho=a) \tag{9.55}$$

Equation (9.55) is called Pocklington's[2] integro-differential equation. It is used to compute the equivalent filamentary current distribution $I_z(z')$ by knowing the incident field on the dipole's surface.

[2] H.C. Pocklington, "Electrical oscillation in wires", *Camb. Phil. Soc. Proc.*, **9**, 1897, pp.324-332.

When the gap of length b is the only place where E_z^i exists, equation (9.55) is written as:

$$\int_{-l/2}^{l/2} I_z(z')\left(\beta^2 G(z,z') + \frac{d^2G(z,z')}{dz^2}\right)dz' = \begin{cases} -j\omega\varepsilon E_z^i, & -\dfrac{b}{2} < z < \dfrac{b}{2} \\ 0, & \dfrac{b}{2} < |z| < \dfrac{l}{2} \end{cases} \quad (9.56)$$

If one assumes that the wire is very thin, then the Green's function $G(z,z')$ simplifies to:

$$G(z,z') = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-j\beta R}}{4\pi R}d\varphi' = \frac{e^{-j\beta R}}{4\pi R}, \quad (9.57)$$

since R reduces to $R_{(\rho=a\to 0)} = \sqrt{4a^2\sin^2\left(\dfrac{\varphi'}{2}\right) + (z-z')^2} \approx z - z'$. Richmond[3] has differentiated and rearranged (9.55), where (9.57) is assumed, in a more convenient for programming form:

$$\int_{-l/2}^{+l/2} I_z(z')\frac{e^{-j\beta R}}{4\pi R^5}\left[(1+j\beta R)\left(2R^2 - 3a^2\right) + \left(\beta a R\right)^2\right]dz' = -j\omega\varepsilon E_z^i \quad (9.58)$$

Equation (9.58) is solved numerically by the Method of Moments, after the structure is discretized into small linear segments.

[3] J.H. Richmond, "Digital computer solutions of the rigorous equations for scattering problems," *Proc. IEEE*, **53**, pp.796-804, August 1965.

4. Hallén's equation

Hallén's[4] equation can be derived as a modification of Pocklington's equation. It is easier to solve numerically, but it makes some additional assumptions. Consider again equation (9.56). It can be written in terms of A_z explicitly as:

$$\frac{d^2 A_z}{dz^2} + \beta^2 A_z = \begin{cases} -j\omega\varepsilon\mu E_z^i, & -\frac{b}{2} < z < \frac{b}{2} \\ 0, & \frac{b}{2} < |z| < \frac{l}{2} \end{cases} \tag{9.59}$$

When $b \to 0$, one can express the incident field in the gap via the voltage applied to the gap:

$$V_g = \lim_{b\to 0} bE_z^i \tag{9.60}$$

The $E_z^i(z)$ function is an impulse function of z, such that:

$$E_z^i = V_g \delta(z) \tag{9.61}$$

The excitation term in (9.59) collapses into a δ-function:

$$\frac{d^2 A_z}{dz^2} + \beta^2 A_z = -j\omega\varepsilon\mu V_g \delta(z) \tag{9.62}$$

If $z \neq 0$,

$$\frac{d^2 A_z}{dz^2} + \beta^2 A_z = 0 \tag{9.63}$$

Because the current density on the cylinder is symmetrical with respect to z', i.e. $J_z(z') = J_z(-z')$, the potential A_z must also be symmetrical. Then, the general solution of the ODE in (9.63) has the form:

$$A_z(z) = B\cos\left(\beta z\right) + C\sin\left(\beta\,|\,z\,|\right) \tag{9.64}$$

From (9.62) it follows that

$$\frac{dA_z}{dz}\Big|_{0_-}^{0^+} = -j\omega\mu\varepsilon V_g \tag{9.65}$$

From (9.64) and (9.65) one can calculate the constant *C*.

$$\frac{dA_z}{dz}\Big|_{0_-}^{0^+} = C\beta\cos(0_+) - C\left(-\beta\right)\cos(0_-) = -j\omega\mu\varepsilon V_g$$

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[4] E. Hallén, "Theoretical investigation into the transmitting and receiving qualities of antennae," *Nova Acta Regiae Soc. Sci. Upsaliensis*, Ser. IV, No. 4, 1938, pp. 1-44.

$$\Rightarrow 2C\beta = -j\omega\mu\varepsilon V_g$$

$$\Rightarrow C = -j\sqrt{\mu\varepsilon}\frac{V_g}{2} = -j\frac{\mu}{\eta}\frac{V_g}{2} \qquad (9.66)$$

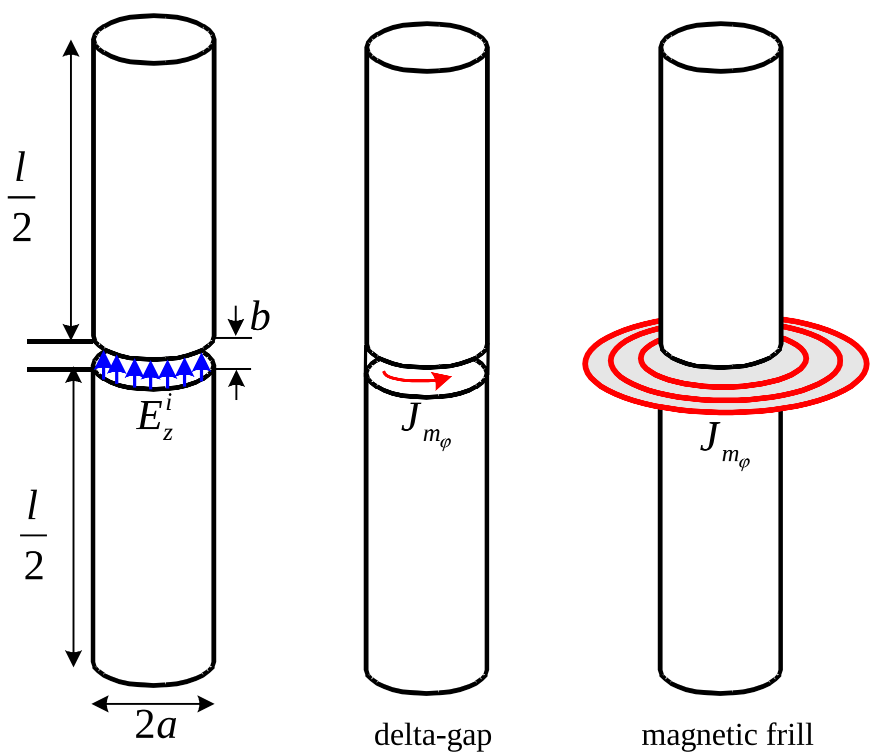
Equation (9.66) is substituted in (9.64), and A_z is expressed with its integral over the currents, to obtain the final form of Hallén's integral equation:

$$\int_{-l/2}^{+l/2} I_z(z')\frac{e^{-j\beta R}}{4\pi R}dz' = -j\frac{V_g}{2\eta}\sin\left(\beta\,|\,z\,|\right) + B\cos(\beta z) \qquad (9.67)$$

Here, $R = \sqrt{a^2 + (z - z')^2}$. It must be repeated again that *Hallén's equation assumes that the incident field exists only in the infinitesimal dipole gap*, while in Pocklington's equation there are no restrictions on the distribution of the incident field at the dipole.

5. Modeling the excitation field

- Delta-gap source (Pocklington and Hallén)
- Magnetic frill source (Pocklington)



delta-gap

magnetic frill